Ph.D. Algebra Exam – May 12, 2009

Answer exactly **seven** of the following eleven problems.

You must show your work. Answers with no work and/or no explanations will receive no credit. State clearly any theorem you use in your proofs.

In the problems, $\mathbf{Q}$ and $\mathbf{C}$ are the fields of rational numbers and complex numbers respectively

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1. Let $\mathcal{C}$ be a category, and denote by **Set** the category of sets. (i) Define what it means for a functor $F : \mathcal{C}^{op} \to \mathbf{Set}$ to be representable. (ii) Show that if X, Y are objects of $\mathcal{C}$ representing the same functor $F : \mathcal{C}^{op} \to \mathbf{Set}$, then $X \simeq Y$.

2. Let R be a ring with identity. Show that the following are equivalent: (i) Every unitary left R-module is projective, (ii) every unitary left R-module is injective, (iii) every exact sequence of unitary left R-modules splits.

3. If G is a group, denote by G^{ab} the abelianization of G, and if X is a set, denote by $F(X)$ and $F^{ab}(X)$ the free group on X and the free abelian group on X respectively. Show that $F(X)^{ab} \simeq F^{ab}(X)$.

4. Let K be a field, $L = K(\alpha)$ a simple extension of K, and $\overline{K}$ an algebraic closure of K. Show that α is separable over K if and only if the ring $L \otimes_K \overline{K}$ has no nilpotent elements.

5. Show that if R is a Noetherian commutative ring with identity, then $R[X]$ is also Noetherian.

6. State and prove Artin's theorem on the linear independence of characters. and use it to prove the multiplicative form of Hilbert's Theorem 90.

7. Let $p(x) = x^7 - 3 \in \mathbf{Q}[x]$, $\zeta \in \mathbf{C}$ a primitive 7-th root of 1, and K the splitting field of $p(x)$. (i) Compute the Galois group of $K/\mathbf{Q}$. (ii) Compute the Galois group of $K/\mathbf{Q}(\zeta)$.

8. (i) Give an example of extensions E/L, L/K where E/K is normal and L/K is not. (ii) Give an example of normal extensions L/K, E/L such that E/K is not normal.

9. Let R be a Dedekind domain, and let I be a non-zero ideal of R. (i) Show that R/I is Artinian. (ii) Show that if a Dedekind domain is Artinian then it is a field.

10. Show that if L/K is any extension of fields, there is a an algebraically independent set $\{\alpha_i\}_{i \in I}$ of elements of L such that L is algebraic over $K(\alpha_i, i \in I)$.

11. Let k be a field. A k-algebra R is *central* if the image of the structure map $k \to R$ is in the center of R. (i) Suppose D is a central k-algebra that is a division algebra (in particular, D is a k-vector space). Show that if k is algebraically closed and D has finite dimension as a k-vector space, then $D = k$.

(ii) Use this result to show that if k is algebraically closed and R is a central k-algebra that is semisimple and of finite dimension over k, then R is a product of matrix algebras over k.