Ph. D. Algebra Exam **May 20, 2008**

Answer seven problems. You should indicate which problems you wish to have graded. Write your answers clearly in complete English sentences. You may quote results (within reason) as long as you state them clearly.

1. Let $n > 2$ and let ζ be a primitive n-th root of unity over $\mathbf{Q}$ (the field of rational numbers). Prove that
$$[\mathbf{Q}(\zeta + \zeta^{-1}) : \mathbf{Q}] = \phi(n)/2.$$

2. Calculate the Galois group of $x^5 - 9x + 3$ over $\mathbf{Q}$, the field of rational numbers. Justify your answer carefully.

3. Prove that the group defined by generators a and b and relations $a^2 = b^3 = 1$, $abab = 1$ is isomorphic to S_3.

4. Let A be a torsion abelian group. Calculate $A \otimes_{\mathbf{Z}} \mathbf{Q}$ where $\mathbf{Q}$ denotes the additive group of the rational numbers.

5. Define *projective* module. Prove from your definition that a module is projective if and only if it is a direct summand of a free module.

6. State and prove Hilbert's Nullstellensatz.

7. Let R be an integral domain. Define what it means for a fractional ideal to be *invertible.* Prove, from your definition, that if I is an invertible fractional ideal of R, then I is finitely generated.

8. Prove the following version of the *Short Five Lemma.* Let R be a ring and let the following diagram of left R-modules
$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \\ 0 & \longrightarrow & A' & \xrightarrow[f']{} & B' & \xrightarrow[g']{} & C' & \longrightarrow & 0 \end{array}$$
be a commutative diagram whose rows are short exact sequences. Suppose that both α and γ are isomorphisms. Prove that β is an isomorphism.

9. Let R be a commutative ring with 1_R. Prove that R is a local ring if and only if for all $r, s \in R$ such that $r + s = 1_R$ either r or s is a unit.

10. Determine up to isomorphism all semisimple *noncommutative* rings with $512 = 2^9$ elements.

11. Let K be a field and let A be a central simple algebra over K such that $\dim_K(A) = n$ is finite. Prove that $A \otimes_K A^{op} \cong \mathrm{M}_{n\times n}(K)$.