PhD Exam in Algebra (May 2007)

Time allowed: Four Hours.

Answer **seven** problems. You may quote results (within reason) as long as you state them clearly.

1. Give the definition of a *normal* algebraic field extension. Let E be a normal algebraic extension of the field F and $g(x) \in F[X]$ an irreducible polynomial. Prove that the irreducible factors of $g(x)$ in $E[X]$ all have the same degree.

2. Determine the Galois group of the polynomial $x^5 - 2 \in \mathbb{Q}[X]$.

3. Define the terms *natural transformation, and natural isomorphism* between functors F and G from a category C to a category D. Show, giving full details, that $M \mapsto R \otimes_R M$ defines a functor from the category of left R-modules to itself, which is naturally isomorphic with the identity functor.

4. Let R be a ring with 1 and

$$0 \to A \to B \to C \to 0$$

be an exact sequence of left unital R-modules.

Show that for any left R-module M, the sequence of induced maps

$$0 \to \mathrm{Hom}_R(C, M) \to \mathrm{Hom}_R(B, M) \to \mathrm{Hom}_R(A, M)$$

is an exact sequence of abelian groups.

5. Prove that the following conditions on a ring R with 1 are equivalent:
(a) Every unital left R-module is projective.
(b) Every short exact sequence of left R-modules splits.
(c) Every unital left R-module is injective.

6. Let R be a commutative ring with 1 and I the intersection of all prime ideals of R. Prove that for each $a \in I$ there exists a natural number n such that $a^n = 0$.

7. Prove that the ring R of $n \times n$ matrices over a division ring is simple. Exhibit a minimal left ideal of R.

8. Define a *primitive ring* and state and prove the Jacobson Density Theorem.

9. Let R be a Noetherian commutative ring with 1. Prove that $R[X]$ is Noetherian.

10. Give examples of the following. Explain your examples, stating clearly any results you use, but you do not need to prove all details.
(a) A Noetherian ring which is not Artinian.
(b) A projective module which is not free.
(c) A Dedekind domain which is not a principal ideal domain.