

Ph. D. Algebra Exam

May 16, 2006

Answer seven problems. You should indicate which problems you wish to have graded. Write your answers clearly in complete English sentences. You may quote results (within reason) as long as you state them clearly.

1. Let  $\mathcal{C}$  be a category.
  - (a) Let  $\{X_\lambda : \lambda \in \Lambda\}$  be a collection of objects in  $\mathcal{C}$ . Give the definition of a coproduct of this collection.
  - (b) Prove that any two coproducts of  $\{X_\lambda : \lambda \in \Lambda\}$  are isomorphic.
  - (c) Prove that coproducts exist in the category of sets.
2. Prove that for each prime  $p$  and each  $n \geq 1$  there is a field with  $p^n$  elements which is determined uniquely up to isomorphism.
3. Let  $E \subset \mathbb{C}$  be the splitting field of  $X^8 - 1$  over  $\mathbb{Q}$ . Determine the Galois group  $G = \text{Gal}(E/\mathbb{Q})$ . For each subgroup  $H \leq G$  determine the fixed field  $E^H$  of  $H$ .
4. Let  $F$  be the free group on two letters. Prove that the commutator subgroup  $[F, F]$  is not finitely generated. Hint: For each  $n \geq 1$  consider the group  $G_n = \langle x_1, \dots, x_n, y : x_i x_j = x_j x_i, y x_i y^{-1} = x_{i+1} \text{ for } 1 \leq i < j \leq n, y x_n y^{-1} = x_1 \rangle$ .
5. Let  $R$  be an integral domain, let  $F$  be the field of fractions of  $R$ , and let  $M, N$  be  $F$ -modules. Prove that  $M \otimes_R N \cong M \otimes_F N$  (as  $R$ -modules).
6. Let  $R$  be a commutative Noetherian ring with 1 and let  $\phi : R \rightarrow R$  be an onto ring homomorphism. Prove that  $\phi$  is one-to-one.
7. Let  $S$  be a commutative ring with 1, let  $R$  be a subring of  $S$  which contains 1, and let  $x \in S$ . Suppose there is a subring  $T \subset S$  such that  $T \supset R[x]$  and  $T$  is finitely generated as an  $R$ -module. Prove that  $x$  is integral over  $R$ .
8. Let  $R$  be a ring and let  $J(R)$  be the Jacobson radical of  $R$ .
  - (a) Prove that every nil left ideal of  $R$  contained in  $J(R)$ .
  - (b) Give an example that shows that  $R$  may contain nilpotent elements which do not lie in  $J(R)$ .
  - (c) Prove that if  $R$  is left Artinian then  $J(R)$  is a nilpotent ideal.
9. Let  $F$  be a field, let  $A$  be a central simple algebra over  $F$ , and let  $B$  be a simple  $F$ -algebra with 1. Prove that  $A \otimes_F B$  is simple.
10. Determine up to isomorphism all semisimple *noncommutative* rings with  $512 = 2^9$  elements.
11. Give an example of a projective module which is not free. Prove that your example has the desired properties.