**Ph. D. Algebra Exam** **August 30, 2005, 6-10 pm**

Answer seven problems. You should indicate which problems you wish to have graded. Write your answers clearly in complete English sentences. You may quote results (within reason) as long as you state them clearly.

1. Let $\mathcal{C}$ be a category and let $\Lambda$ be a set. For each $\lambda \in \Lambda$ let $X_\lambda$ be an object in $\mathcal{C}$.
   (a) Give the definition of a product of the collection of objects $\{X_\lambda\}_{\lambda\in\Lambda}$.
   (b) Prove that any two products of the $\{X_\lambda\}_{\lambda\in\Lambda}$ are isomorphic.
   (c) Prove that products always exist in the category of groups.

2. Show that the alternating group $A_n$ is a simple group for $n \geq 5$. Show that the symmetric group $S_n$ is not solvable for $n \geq 5$.

3. State and prove Jacobson's Density Theorem.

4. Consider the polynomial
   $$p(x) = x^5 - 3x^4 - 2x^2 + 2x + 2$$
   over $\mathbb{Q}$. Can all the roots of $p(x)$ be written by radicals over $\mathbb{Q}$? Can they be written by radicals over $\mathbb{C}$? Carefully justify both answers.

5. Suppose that $F$ is a finite extension of $\mathbb{Q}$. Prove that $F$ contains only a finite number of roots of unity.

6. Let $F$ be the free group in generators $x$ and $y$. Let $N$ be the subgroup of $F$ generated by the set
   $$S = \{z^2 : z \in F\}.$$
   Prove that $N$ is a normal subgroup of $F$, and calculate the order of $F/N$.

7. Let $A$ be a torsion abelian group, and let $\mathbb{Q}$ be the (additive) group of the rational numbers. Prove that $A \otimes_{\mathbb{Z}} \mathbb{Q} = \{0\}$, and that $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} \simeq \mathbb{Q}$.

8. Let $R$ be an integral domain with quotient field $F$. Suppose that $a \in R$ is such that $a \neq 0$ and $1/a$ is integral over $R$. Prove that $a$ is a unit of $R$.

9. Let R be a commutative ring with $1 \neq 0$. Prove that the set consisting of 0 and all zero divisors in R contains at least one prime ideal.

10. Let $R$ be a commutative ring with identity different from zero. Let $F_1$ and $F_2$ be two finitely generated free $R$-modules, and assume that $F_1 \simeq F_2$ as $R$-modules. Prove that the cardinality of the set of free generators for $F_1$ equals the cardinality of the set of generators for $F_2$.

11. Let $R$ and $S$ be semi-simple rings with exactly 400 elements. Assume that neither $R$ nor $S$ is commutative. Does it follow that $R \simeq S$? Justify your answer.