

Ph. D. Algebra Exam

May 17, 2005

Answer seven problems. You should indicate which problems you wish to have graded. Write your answers clearly in complete English sentences. You may quote results (within reason) as long as you state them clearly.

1. Let \mathcal{C} be a concrete category. Define what is meant by a *free object* in \mathcal{C} . Let \mathcal{G} be the category of all finite groups. Prove that the free objects in \mathcal{G} are exactly the trivial groups.
2. State and prove Maschke's Theorem.
3. Let R be a left-Artinian ring such that, for all $r \in R$, we have $r^2 = r$. Prove that R is finite and commutative. Prove that any two such rings are isomorphic to each other if and only if they have the same number of elements.
4. Let F be the splitting field of $p(x) = x^3 + 5x + 5$ over \mathbb{Q} . Prove that there is a unique intermediate field K such that $K \neq \mathbb{Q}$, $K \neq F$, and K/\mathbb{Q} is Galois.
5. State and prove Hilbert's Theorem 90.
6. Let $G = \langle x, y : x^3 = y^2 = 1 \rangle$. Prove that G is infinite and non-abelian.
7. Let A be a finitely generated abelian group. Consider $T = \mathbb{R} \otimes_{\mathbb{Z}} A$. Describe for which abelian groups A we have that T is not trivial. Prove your description. In your proof, you may assume the existence and uniqueness of the tensor product as an abelian group, but you need to prove any other property of the tensor product that you use.
8. Prove the *Lying-Over Theorem*: Let S be an integral extension of an integral domain R , and let P be a prime ideal of R . Then, there exists a prime ideal Q of S , such that $Q \cap R = P$.
9. Define what is a *Dedekind domain*. Give an example of a Dedekind domain which is not a unique factorization domain.
10. Prove that an invertible ideal in an integral domain that is a local ring is principal.
11. Let R be a ring with identity. Consider the ring $\text{Hom}_R(R, R)$ of left R -module homomorphisms from R to itself. Prove that $\text{Hom}_R(R, R)$ is isomorphic, as a ring, to R^{op} .