Ph. D. Algebra Exam **September 9, 2004**

Answer seven problems. You should indicate which problems you wish to have graded. Write your answers clearly in complete English sentences. You may quote results (within reason) as long as you state them clearly.

1. (a) Define concrete category.
   (b) Define what it means for an object F in a concrete category to be free on a set S.
   (c) Given a set S, construct a free object on S in the category of abelian groups.
   (d) Given a set S, construct a free object on S in the category of sets.
2. Let F be a field, let A be a central simple algebra over F, and let B be a simple F-algebra with 1. Prove that $A \otimes_F B$ is simple.
3. Use the Wedderburn-Artin Theorem to determine all isomorphism classes of semisimple rings of order $720 = 2^4 \cdot 3^2 \cdot 5$.
4. Let $\zeta = e^{2\pi i/16}$ be a primitive 16th root of unity and let $F = \mathbb{Q}(\zeta)$. Determine the Galois group $G = \mathrm{Gal}(F/\mathbb{Q})$, and for each subgroup $H \leq G$ determine the fixed field F^H.
5. Let p be prime and let n be a positive integer. Prove that there is a field $\mathbb{F}_{p^n}$ with p^n elements, and that this field is uniquely determined up to isomorphism.
6. Let $G = \langle x, y : x^3 = y^2,\ y^4 = 1\rangle$. Prove that G is infinite and non-abelian.
7. Give an example of a projective module which is not free. Justify your answer.
8. Let R be an Artinian commutative ring with 1. Prove that R has only finitely many prime ideals, all of which are maximal.
9. Let R be a commutative ring with 1 and let S be a multiplicative subset of R which contains no zero divisors. Prove that there is a one-to-one correspondence between prime ideals of $S^{-1}R$ and prime ideals P of R such that $P \cap S = \emptyset$.
10. Let R be a commutative ring with 1.
    (a) Prove that if I_1 and I_2 are ideals in R such that $I_1 \cap I_2 = P$ is prime then either $I_1 = P$ or $I_2 = P$.
    (b) Is the statement above still true if "prime" is replaced by "primary"? Justify your answer.
11. Let $R \subset S$ be commutative rings with $1_R = 1_S$ such that S is integral over R. Let C be an algebraically closed field and let $\phi : R \to C$ be a unitary ring homomorphism. Prove that there is a ring homomorphism $\tilde{\phi} : S \to C$ such that $\tilde{\phi}|_R = \phi$.