Ph. D. Algebra Exam **May 13, 2004**

Answer seven problems. You should indicate which problems you wish to have graded. Write your answers clearly in complete English sentences. You may quote results (within reason) as long as you state them clearly.

1. Let F be a field and let E be a finite extension of F. Say that E/F is a simple extension if there is $\alpha \in E$ such that $E = F(\alpha)$.
   (a) Prove that E/F is a simple extension if and only if there are only finitely many many fields K such that $E \supset K \supset F$.
   (b) Prove that if F has characteristic 0 then E/F is a simple extension.

2. Let p and q be primes and let $E \subset \mathbb{C}$ be the splitting field for $g(X) = X^p - q$ over $\mathbb{Q}$.
   (a) Give generators for E as an extension of $\mathbb{Q}$.
   (b) Determine the degree of the extension $E/\mathbb{Q}$.
   (c) Give generators and relations for $\mathrm{Gal}(E/\mathbb{Q})$.
   (d) Describe how the generators of $\mathrm{Gal}(E/\mathbb{Q})$ act on the generators of E.

3. Let G be a finite nilpotent group, i. e., a group whose ascending central series has the form
$$1 = Z_0(G) \subsetneq Z_1(G) \subsetneq \ldots \subsetneq Z_{n-1}(G) \subsetneq Z_n(G) = G$$
for some $n \geq 0$. Prove that the descending central series $G = G^0 \geq G^1 \geq G^2 \geq \ldots$ also has length n, and that $G^i \leq Z_{n-i}(G)$ for $0 \leq i \leq n$.

4. Let R be a ring with 1.
   (a) Prove that if J is a divisible abelian group then $\mathrm{Hom}_{\mathbb{Z}}(R, J)$ is a divisible left R-module.
   (b) Prove that if M is a unitary left R-module then M is isomorphic to a submodule of a divisible left R-module.

5. Let $\mathcal{C}$ be a category and let Λ be a set. For each $\lambda \in \Lambda$ let X_λ be an object in $\mathcal{C}$.
   (a) Give the definition of a coproduct of the collection of objects $\{X_\lambda\}_{\lambda\in\Lambda}$.
   (b) Prove that any two coproducts of the $\{X_\lambda\}_{\lambda\in\Lambda}$ are isomorphic.
   (c) Prove that coproducts always exist in the category of abelian groups.

6. Let F be a field and let $F[[X]]$ denote the ring of formal power series over F.
   (a) Prove that $a_0 + a_1X + a_2X^2 + \cdots \in F[[X]]$ is a unit if and only if $a_0 \neq 0$.
   (b) Prove that $F[[X]]$ is a discrete valuation ring.

7. Let R be a commutative Noetherian ring with 1 and let $I \subset R$ be an ideal. Prove that I has a primary decomposition.

8. Let R be a left Artinian ring. Prove that the following are equivalent:

   (a) R is simple.

   (b) R is primitive.

   (c) $R \cong M_n(D)$ for some $n \geq 0$ and division ring D.

9. Let R be a ring and let M be a left R-module. Assume that for every nonzero $x \in M$ we have $Rx \neq 0$, and that for every submodule $N \subset M$ there is a submodule N' such that $M = N \oplus N'$. Prove that M is generated by its simple submodules.

10. (a) State the Noether-Skolem theorem.

    (b) Let D be a division ring such that

       i. $\mathbb{R}$ is a subfield of D.

       ii. Every $\alpha \in D$ is algebraic over $\mathbb{R}$.

    Prove that D is isomorphic to either $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$.

    (Here $\mathbb{R}$ is the real numbers, $\mathbb{C}$ is the complex numbers, and $\mathbb{H}$ is Hamilton's quaternions.)

11. Let $m, n \geq 1$. Describe the ring $(\mathbb{Z}/m\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z})$. In particular, what is the cardinality of this ring?