Ph.D. Algebra Exam – September 2003

Time allowed: 240 minutes

Do **seven** of the following eleven problems. Do not turn in more than seven problems. You must show your work. Answers with no work and/or no explanations will receive **no** credit. State clearly any theorem you use in your proofs.

In the problems, **Z**, resp. **Q**, **C**, is the set of all integers, resp. of all rational numbers, of all complex numbers.

1. Consider the following statement $A(P)$: "*If a normal subgroup H of a group G and the quotient G/H both have property P, then so does G.*" Prove or disprove $A(P)$, where
a) P is "being solvable";
b) P is "being nilpotent".

2. Inside the symmetric group S_7, consider the subgroup G generated by the permutations (1234567) and (235)(476). Let

$$H = \langle a, b \mid a^7 = b^3 = 1, bab^{-1} = a^2 \rangle .$$

Show that $G \simeq H$.

3. Let G be a finite group with exactly one maximal subgroup. Prove that G is cyclic of prime power order.

4. Let $\mathbf{F}$ be a finite field of finite cardinality q and n be any natural number. Show that there is at least one irreducible polynomial $f \in \mathbf{F}[x]$ of degree n.

$A \neq 0$

5. Let A be a commutative ring with identity. Suppose that M and N are free A-modules with m and n generators, respectively. Prove that if $M \simeq N$ then $m = n$.

6. a) Give an example of a projective **Z**-module and an example of an injective **Z**-module.
b) Let A be a finite abelian group considered as a **Z**-module and let I be an injective **Z**-module. Compute $A \otimes_{\mathbf{Z}} I$.

7. Let $\alpha \in \mathbf{C}$ be such that $[\mathbf{Q}(\alpha) : \mathbf{Q}] = 2003$. Let $\mathbf{E}/\mathbf{Q}$ be a normal closure of $\mathbf{Q}(\alpha)/\mathbf{Q}$. Prove that $[\mathbf{E} : \mathbf{Q}]$ divides 2003! .

8. Let $\xi \in \mathbf{C}$ be a primitive 77^{th} root of unity in $\mathbf{C}$, and let $\mathbf{F} = \mathbf{Q}(\xi)$.
a) Briefly explain why $\mathbf{F}/\mathbf{Q}$ is a Galois extension, and describe the structure of the Galois group $Aut_{\mathbf{Q}}(\mathbf{F})$.
b) Find the number of subfields of $\mathbf{F}$ that are quadratic extensions of $\mathbf{Q}$.

9. State and prove the Hilbert Basis Theorem.

10. Let D be a Dedekind domain and F be the group of fractional ideals of D. Prove that any nontrivial element in F has infinite order.

11. Let R be a commutative ring with identity such that every element $x \in R$ satisfies $x^{n(x)} = x$ for some integer $n(x) \geq 1$ depending on x. Prove that the Jacobson radical of R is 0.