Ph.D. Algebra Exam – January 2003

Time allowed: 240 minutes

Do **seven** of the following eleven problems. Please do not turn in more than seven problems.

You must show your work. Answers with no work and/or no explanations will receive **no** credit. State clearly any theorem you use in your proofs.

In the problems, $\mathbf{Z}$, resp. $\mathbf{Q}$, $\mathbf{C}$, is the set of all integers, resp. of all rational numbers, of all complex numbers.

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1. State and prove the Jordan-Hölder Theorem for finite groups.

2. Let F be the free group in two generators a and b. Let N be the smallest normal subgroup of F which contains a^5, b^2 and $abab$. Prove that F/N is a non-abelian group of order 10.

3. Let G be an abelian group. Define the torsion subgroup G_t of G. Prove from your definition that G_t is a subgroup of G and that G/G_t is torsion free.

4. Let ζ be a primitive 2003-rd root of unity. Prove that $\mathbf{Q}(\zeta)$ is a Galois extension of $\mathbf{Q}$. Calculate the Galois group of $\mathbf{Q}(\zeta)/\mathbf{Q}$ and the degree $[\mathbf{Q}(\zeta) : \mathbf{Q}]$ of the extension. Furthermore, calculate the degree $[\mathbf{Q}(\zeta) \cap \mathbf{R} : \mathbf{Q}]$. (Hint: 2003 is a prime).

5. A commutative ring R with identity is said to be Noetherian if every ideal of R is finitely generated. Let R be a Noetherian ring, and let $\mathcal{C}$ be an ascending chain of ideals of R. Prove that $\mathcal{C}$ only contains a finite number of distinct ideals.

6. Prove the following version of the Lying-Over Theorem. Let S be an integral extension ring of a commutative ring R with identity. Let P be a prime ideal of R. Prove that there exists some prime ideal Q of S such that $Q \cap R = P$.

7. Let $A = \mathbf{Z}/2\mathbf{Z}$ and $B = \mathbf{Z}/3\mathbf{Z}$ be viewed as $\mathbf{Z}$-modules. Calculate the $\mathbf{Z}$-module $A \otimes B$, and, in particular, give the cardinality of $A \otimes B$.

8. Among the following integral domains, decide which ones are Dedekind domains, and give a brief explanation.
   (a) $\mathbf{Z}[T]$, the polynomial ring over the integers, in one variable.
   (b) $\mathbf{Z}[\sqrt{-5}] = \{\, a + b\sqrt{-5} \, : \, a, b \in \mathbf{Z} \,\}$.
   (c) The ring $k[[T]]$ of all formal power series in one variable, over the field k.
   (d) $k[T_1, T_2]$, the polynomial ring in two variables, over the field k.

9. Give one representative for each conjugacy class of elements of order 4 in $\mathrm{GL}(4, \mathbf{Q})$.

10. Calculate the number of irreducible polynomials of degree 4 over the field of 2 elements.

11. Let M be a semi-simple module over a ring R. Prove that every homomorphic image of M is semi-simple.