Ph.D. Algebra Exam – May 2002

Time allowed: 240 minutes

Do **seven** of the following eleven problems. Please do not turn in more than seven problems.

You must show your work. Answers with no work and/or no explanations will receive **no** credit. State clearly any theorem you use in your proofs.

In the problems, $\mathbf{Z}$, resp. $\mathbf{Q}$, $\mathbf{C}$, is the set of all integers, resp. of all rational numbers, of all complex numbers.

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1. Let X be a set and let $\mathcal{C}$ be a concrete category. Define what is meant for an object F together with a set map $i : X \to F$ to be *free*. Prove that if (F, i) and (F', i') are free for the same set X, then F and F' are equivalent (isomorphic) in the category $\mathcal{C}$.

2. Let R be a ring and let P be a left R-module. Define what is meant for P to be *projective*. Prove, from your definition, that a module is projective if and only if it is isomorphic to a direct summand of a free module.

3. Define what is meant for a group to be *solvable*. Prove that the homomorphic image of every solvable group is solvable. Prove that every subgroup of a solvable group is solvable.

4. Suppose R is a commutative ring with identity. Let M and N be R-modules, and let $\phi : M \to N$ be an injective module homomorphism. Let P be a projective R-module

- Show that $\mathrm{Id}_P \otimes \phi : P \otimes_R M \to P \otimes_R N$ is injective.
- If Q is an arbitrary R-module is $\mathrm{Id}_Q \otimes \phi : Q \otimes_R M \to Q \otimes_R N$ is necessarily injective? Justify your answer.

5. State and prove Hilbert's Basis Theorem.

6. State and prove Hilbert's Nullstellensatz.

7. Let $p(x) = x^7 - 3 \in \mathbf{Q}[x]$, and let $\zeta \in \mathbf{C}$ be a primitive 7-th root of 1.
a) Compute the Galois group of $p(x)$ over $\mathbf{Q}$.
b) Compute the Galois group of $p(x)$ over $\mathbf{Q}(\zeta)$.

8. Let F be a finite field with exactly 4 elements. How many elements does the splitting field of the polynomial $x^6 - 1 \in F[x]$ have? Justify your answer.

9. Let R be a Dedekind domain, and let I be a non-zero ideal of R. Show that R/I is Artinian. Are all Dedekind domains Artinian? Justify your answers.

10. How many conjugacy classes of elements of order 3 are there in the general linear group $\mathrm{GL}(4, \mathbf{Q})$? Justify your answer.

11. Prove or disprove: Let R and S be semi-simple rings with $|R| = |S| = 2002 = 2 \cdot 7 \cdot 11 \cdot 13$. Then R is isomorphic to S.