Ph.D. Algebra Exam – May 2001

Time allowed: 240 minutes

Do **seven** of the following eleven problems. Please do not turn in more than seven problems.

You must show your work. Answers with no work and/or no explanations will receive **no** credit. State clearly any theorem you use in your proofs.

In the problems, **Z**, resp. **Q**, **C**, is the set of all integers, resp. of all rational numbers, of all complex numbers.

1. Let G be a finite group of order $m \cdot n$ and K a normal subgroup of G of order m, where m and n are coprime.

(i) Prove that G has exactly one subgroup of order m.

(ii) Suppose that n is a prime power. Is there any subgroup H of G such that G is a semidirect product of K by H ? Justify your answer.

2. Prove that any finite group of order $616 = 11 \cdot 8 \cdot 7$ is solvable. You may assume that every group of order $p^a q^b$, where p and q are primes, is solvable.

3. Let $GL_2(\mathbf{C})$ be the group of invertible 2×2 complex matrices under matrix multiplication and let G be a finite subgroup of $GL_2(\mathbf{C})$. Suppose that G is simple and $|G|$ is even. Prove that $|G| = 2$. (**Hint** : Consider elements of order 2 of G.)

4. Let A be a 3×3 complex matrix. Suppose that $Tr(A) = Tr(A^2) = Tr(A^3) = 0$ (where $Tr(X)$ is the trace of the square matrix X). Prove that $A^{2001} = 0$.

5. Let q be a prime power and $\mathbf{F}$ a field of q elements. Find the number $N(q)$ of monic irreducible polynomials of degree 2 over $\mathbf{F}$.

6. Let p be a prime. Let $\mathbf{K}$ be a subfield of $\mathbf{C}$ with the property:

"If $\mathbf{E}$ is a finite extension of $\mathbf{K}$ such that $[\mathbf{E} : \mathbf{K}]$ is coprime to p, then $\mathbf{E} = \mathbf{K}$."

Prove that if $\mathbf{F}$ is any finite extension of $\mathbf{K}$ then $[\mathbf{F} : \mathbf{K}]$ is a p-power. (**Hint** : Use normal closure, Galois Theory, and Sylow's Theorem).

7. Let $\xi \in \mathbf{C}$ be such that $\xi^{2001} = 5$. Show that $\mathbf{Q}(\xi)$ cannot be contained in any cyclotomic extension of $\mathbf{Q}$.

8. Compute the integral closure R of $\mathbf{Z}$ in $\mathbf{Q}(\sqrt{-5})$. Is R a unique factorization domain? A principal ideal domain ? A Dedekind domain ? Justify your answer(s).

9. Let R be a local ring. Prove that every finitely generated projective R-module is free.

10. State and prove the Hilbert Nullstellensatz.

11. Let R be a primitive ring such that $ab - ba$ is nilpotent for all $a, b \in R$. Prove that R is a division ring.