Ph.D. Algebra Exam – January 2001

Time allowed: 240 minutes

Do **seven** of the following eleven problems. Please do not turn in more than seven problems.

You must show your work. Answers with no work and/or no explanations will receive **no** credit. State clearly any theorem you use in your proofs.

In the problems, **Z**, resp. **Q**, **C**, is the set of all integers, resp. of all rational numbers, of all complex numbers.

1. Let G be a finite simple (abelian or non-abelian) group of order n. Find the number of normal subgroups of $G \times G$.

2. Show that there is no simple group of order 112. (**Caution** : You are not allowed to use Burnside's $p^a q^b$-Theorem.)

3. Let N be a minimal normal subgroup of a finite group G. Assume N is solvable. Show that N is an elementary abelian p-group for some prime p.

4. Let A, B be two matrices over a field **F**. Suppose that $\begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}$ is similar to $\begin{pmatrix} B & 0 \\ 0 & B \end{pmatrix}$. Prove that A is similar to B.

5. Determine the Galois group over **Q** of
a) $(x^3 - 2)(x^2 - 3)$;
b) $(x^3 - 2)(x^2 + 3)$.

6. Is there any integer d such that: for any separable extension **F** of degree 5 of a field **K**, if an extension **E** of **K** is a normal closure of **F**/**K** then $[\mathbf{E} : \mathbf{K}] \leq d$? If yes, prove your bound. If not, justify why.

7. Let $\xi \in \mathbf{C}$ be such that $\xi^{2000} = 3$. Show that -3 is not a sum of squares in $\mathbf{Q}(\xi)$.

8. Prove the Hilbert Basis Theorem: *If R is a commutative Noetherian ring with identity, then so is $R[x_1, x_2, \ldots, x_n]$.*

9. Let A be a finite dimensional (not necessarily commutative) **C**-algebra with no zero divisors, but with identity. Show that $\dim_{\mathbf{C}} A = 1$.

10. a) If the ring R is a principal ideal domain, prove that every nonzero prime ideal in R is maximal.
b) Is the statement in a) still true if R is not a principal ideal domain? (**Hint** : Consider the ring $\mathbf{Z}[x]$.)

11. Suppose that R is a ring with identity.
a) Prove that each free left R-module is projective.
b) Prove that a left R-module P is projective if and only if each short exact sequence of left R-modules

$$0 \longrightarrow A \longrightarrow B \longrightarrow P \longrightarrow 0$$

splits.