PhD Algebra Examination; September, 2000

Work exactly seven out of the following eleven exercises.

1. (a) Define *solvable group.*

   (b) Prove that the homomorphic image of a solvable group is solvable.

   (c) Prove that a free group is solvable if and only if it is the free group on at most one generator.

2. Let G be a group; call $g \in G$ a *non-generator* if, for each subset X of G so that $X \cup \{g\}$ generates G, then, in fact, X itself generates G. Let $\mathrm{Fr}(G)$ denote the set of all non–generators of G.

   (a) Prove that $\mathrm{Fr}(G)$ is a subgroup of G.

   (b) Show that $\mathrm{Fr}(G)$ is the intersection of all maximal (proper) subgroups of G. (Careful with Zorn's Lemma!)

3. Suppose that R is a principal ideal domain. Prove that any submodule of a free R–module is free.

4. Suppose $(m, n) = 1$. Compute $\mathbb{Z}_m \otimes_{\mathbb{Z}} \mathbb{Z}_n$. Justify your answer.

5. Let R be a finite semisimple ring with identity. Suppose that no fourth power $k^4 > 1$ ($k \in \mathbb{N}$) divides $|R|$. Prove that R is commutative, and therefore a direct product of fields.

6. Suppose that R is a ring with identity. Prove that

$$\mathrm{Hom}_{\mathbf{Ab}}(B, \prod_{i \in I} G_i) \cong \prod_{i \in I} \mathrm{Hom}_{\mathbf{Ab}}(B, G_i),$$

   as right R–modules, for all left R–modules B and all abelian groups G_i ($i \in I$). **Ab** denotes the category of all abelian groups together with all homomorphisms between them.

   You may use resources from category theory; if so, outline your argument so that it is clear which theorems you are appealing to.

7. Prove Nakayama's Lemma: let A be a commutative ring with identity. Let M be a finitely generated A–module, and I be an ideal of A, contained in the Jacobson radical $J(A)$ of A. Show that if $IM = M$ then $M = \{0\}$.

8. Let A be a commutative ring with identity. For each multiplicative system S of A, prove that $S^{-1}A$ is a flat A–module.

9. Suppose that A is a subring of the commutative ring B with 1. If B is integral over A, prove that every homomorphism f of A into the algebraically closed field L admits an extension to a homomorphism $g : B \longrightarrow L$.

10. Let Θ_p be the p–th cyclotomic polynomial over the field $\mathbb{Q}$, where p is a prime number. Show that the Galois group of the splitting field of Θ_p over $\mathbb{Q}$ is cyclic of order $p-1$. (Be clear about any theorems you quote.)

11. Consider the polynomial over the field $\mathbb{F}_2$ of two elements: $g(x) = x^4 + x^3 + x^2 + x + 1$.

    (a) Prove that $g(x)$ is irreducible over $\mathbb{F}_2$.

    (b) Let K be a splitting field for $g(x)$ over $\mathbb{F}_2$, and let $r \in K$ be a root of $g(x)$. Factor $g(x)$ into irreducibles over $\mathbb{F}_2(r)$.

    (c) Show that $K = \mathbb{F}_2(r)$.

    (d) Find the Galois group $\mathrm{Gal}(K/\mathbb{F}_2)$.