PhD Algebra Examination; May, 2000

Please work out seven out of the following eleven exercises. Please do not hand in more than seven.

1. Let F denote the free group on the set X. Prove that $F/[F,F]$ is the free abelian group on the set X. (Note: $[F,F]$ denotes the commutator subgroup of F.)

2. Recall that a subgroup H of a product of groups $G = \prod_{i\in I} G_i$ is said to be a *subdirect product* if, for each $i \in I$, the restriction of the projection $\pi_i : G \longrightarrow G_i$ to H is surjective.

   Let $\mathcal{V}$ be a class of groups which is closed under the formation of subdirect products and homomorphic images. Prove that $\mathcal{V}$ is also closed under formation of subgroups.

3. Let R be a ring with identity, and M be a left R–module. A *projective basis* for M is a subset $\{ a_i : i \in I \}$ of M together with R–homomorphisms $f_i : M \longrightarrow R$ such that

   (i) for each $a \in M$, $|\{ i \in I : f_i(a) \neq 0 \}| < \infty$, and

   (ii) for each $a \in M$, $a = \sum_{i\in I} f_i(a)a_i$.

   Prove that M is a projective module if and only if it has a projective basis.

4. Suppose that G is a finite group and K is a field. Prove:

   (a) If the characteristic of K does not divide $|G|$, then $K[G]$, the group algebra over K, is a semisimple left Artinian ring. (You may use that $K[G]$ is a ring with identity, and the Wedderburn–Artin Theorem in all its glory.)

   (b) Show that if the characteristic of K does divide the order of G, then the Jacobson radical of $K[G]$ is nontrivial.

5. Let A be a commutative ring with identity, and suppose that M is an A–module, and I is an ideal of A.

   (a) Prove that
   $$(A/I) \otimes M \cong M/IM,$$
   where IM is the submodule generated by all elements of the form xb, with $x \in I$, $b \in M$.

   (b) Prove that the isomorphism in (a) is natural.

6. Suppose that A is a commutative ring with identity. If every prime ideal of A is finitely generated, show that A is Noetherian.

7. State and prove the Hilbert Basis Theorem.

8. Let K be a field.

   (a) Define *discrete valuation* on K, and the notion of a *discrete valuation ring.*

   (b) Suppose that A is an integral domain. Prove that it is a discrete valuation ring if and only if it is a principal ideal domain in which
   $$p \text{ and } q \text{ irreducible } \Rightarrow\ p = uq, \text{ for some unit } u \in A.$$

9. Prove that the lattice of all ideals of a Dedekind domain is distributive. (Hint: Localize!)

10. Use the notion of formal derivatives to show that, if $f(T)$ is an irreducible polynomial over the field F then it has repeated roots in the splitting field if and only if the characteristic of F is $p > 0$, and $f(T) = g(T^p)$, for some $g(T) \in F[T]$.

11. Let p be a prime number.

    (a) Prove that if a subgroup H of S_p, the symmetric group on p letters, contains a p-cycle and a transposition, then $H = S_p$.

    (b) If $p(T)$ is an irreducible polynomial over $\mathbb{Q}$, of degree p, having exactly two non-real roots, then show that the Galois group of the splitting field of $p(T)$ is S_p.