SUGGESTED PHD ALGEBRA EXAM
(SEPTEMBER 1999) - PRELIMINARY VERSION

Please do 7 out of the 11 problems below.

Question 1. Here, $\mathbb{Z}_n$ denotes the cyclic group of order n.
(a). Let p be a prime. How many subgroups of order p^2 does the abelian group $\mathbb{Z}_{p^3} \oplus \mathbb{Z}_{p^2}$ have?
(b). What are the elementary divisors of the group $\mathbb{Z}_{26} \oplus \mathbb{Z}_{42} \oplus \mathbb{Z}_{49} \oplus \mathbb{Z}_{200}$? What are its invariant factors?
(c). Determine up to isomorphism all abelian groups of order 20.

Question 2.
(a). Exhibit an automorphism of the cyclic group of order 6 that is not an inner automorphism.
(b). Show that if a normal subgroup N of order p (p prime) is contained in a group G of order p^n, then N is in the center of G.

Question 3.
(a). Is the additive group $\mathbb{Q}$ of rationals a divisible abelian group?
(b). Show that no nonzero finite abelian group is divisible.
(c). Show that no nonzero free abelian group is divisible.

Question 4. Let R and S be rings and A_R, ${}_RB_S$, C_S (bi)-modules. Demonstrate an isomorphism of abelian groups between $\mathrm{Hom}_S(A \otimes_R B, C)$ and $\mathrm{Hom}_R(A, \mathrm{Hom}_S(B, C))$. (You should state very clearly what must be checked.)

Question 5. Let R be a ring, let

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

be a short exact sequence of left R-modules and let D be a right R-module. Show that

$$0 \longrightarrow D \otimes_R A \xrightarrow{1_D \otimes f} D \otimes_R B \xrightarrow{1_D \otimes g} D \otimes_R C \longrightarrow 0$$

is a short exact sequence of abelian groups under each one of the following hypothesis:

(a). $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ is split exact.
(b). R has an identity and D is a free right R-module.
(c). R has an identity and D is a projective unitary right R-module.

Question 6. Let A be a commutative ring with 1 such that every element $x \in A$ satisfies $x^n = x$ for some $n > 1$, depending on x. Prove that

(a). every prime ideal of A is maximal.
(b). the Jacobson radical of A is (0).

Question 7. Let $\mathcal{C}$ and $\mathcal{D}$ be categories, and let S and T be covariant functors from $\mathcal{C}$ to $\mathcal{D}$.
(a). Define a *natural isomorphism* $\alpha : S \to T$.
(b). If B is a unitary left module over a ring R with identity, show that there is a "natural" isomorphism of modules

$$\alpha_B : R \otimes_R B \cong B.$$

Question 8.
(a). Give an example of commutative rings R, A with $R \subset A$, where A is Noetherian but R is not.
(b). Let $R = K[x, y]$ be the polynomial ring in two variables over the field K, let $\mathcal{A}$ denote the ideal (xy, y^2) in R and let $c \in K$. Show that

$$\mathcal{A} = (y) \cap (x + cy, y^2)$$

and that this is a primary decomposition.

Question 9.
(a). Show that if S is a multiplicative subset of a Dedekind domain R (with $1 \in S$, $0 \notin S$), then $S^{-1}R$ is a Dedekind domain.
(b). Show that if I is a nonzero ideal in a Dedekind domain, then R/I is an Artinian ring.

Question 10. Let p be a prime and $n \geq 1$ an integer. Show that F is a finite field with p^n elements if and only if F is a splittting field of $(x^{p^n} - x)$ over the field $\mathbb{Z}_p$ of p elements.

Question 11.
(a). Give an example of fields $M \supset L \supset K$ such that M/L and L/K are normal extensions, but M/K is not normal. Justify your answer.
(b). Let $L \supset K$ be fields such that $[L : K] = 8$. Prove that there exist α, β, $\gamma \in L$ such that $L = K(\alpha, \beta, \gamma)$.