PHD ALGEBRA EXAM (MAY 1999) — VERSION 3

Please do 7 out of the 11 problems below.

Question 1. An algebraic extension field F of K is said to be **normal** over K if every irreducible polynomial in $K[x]$ that has a root in F actually splits in $F[x]$.

Prove that an algebraic extension F of K is normal over K if and only if for every irreducible $f \in K[x]$, f factors in $F[x]$ as a product of irreducible factors all of which have the same degree.

Question 2.
(a). Show that the additive group of rationals $\mathbb{Q}$ is not free.
(b). Show that the group $\mathbb{Q}^*$ of all positive rationals (under multiplication) is free abelian with basis $\{p : p \text{ is prime in } \mathbb{Z}\}$.

Question 3. Show that any simple group G of order 60 is isomorphic to A_5. (If you want, you may use the fact that for each $n \geq 2$, A_n is the only subgroup of S_n of index 2.)

Question 4. Let R be an integral domain and for each maximal ideal M, consider the localization R_M as a subring of the quotient field of R. Show that $\cap R_M = R$, where the intersection is taken over all maximal ideals M of R.

Question 5.
(a). Define solvable group.
(b). Prove that any group of order 48 is solvable (without using Burnside's (p,q)-Theorem).

Question 6. Let R be a ring. Show that the following conditions on an R-module P are equivalent.

(i). P is projective.
(ii). Every short exact sequence

$$0 \to A \xrightarrow{f} B \xrightarrow{g} P \to 0$$

is split exact.
(iii). There is a free module F and an R-module K such that $F \cong K \oplus P$.

Question 7. Let $\mathcal{C}$ and $\mathcal{D}$ be categories, and let S and T be covariant functors from $\mathcal{C}$ to $\mathcal{D}$.
(a). Define a *natural isomorphism* $\alpha : S \to T$.
(b). If B is a unitary left module over a ring R with identity, show that there is a "natural" isomorphism of modules

$$\alpha_B : R \otimes_R B \cong B.$$

Question 8. Let R be a commutative ring with identity.
(a). Let $\mathcal{A}$ be an ideal in R, and assume that M is a finitely generated R-module such that $\mathcal{A}.M = M$. Show that there is some $a \in \mathcal{A}$ satisfying $(1+a)M = 0$.
(b). Let M be a finitely-generated R-module. Show that $J(R).M = M$ implies $M = \{0\}$. (Here, $J(R)$ denotes the Jacobson radical of R.)

Question 9.
(a). Show that if R is a unique factorization domain, then R is integrally closed.
(b). Find the integral closure of $\mathbb{C}[x^5, x^7]$.

Question 10.
(a). Show that $\mathbb{C}[x,y]/(xy-1)$ (quotient of polynomial ring in 2 variables) is not isomorphic to $\mathbb{C}[t]$ (polynomial ring in 1 variable).
(b). Show that the set

$$\{(m,n) : m,n \in \mathbb{Z}\}$$

is not an algebraic variety in $\mathbb{C}^2$.

Question 11.
(a). Find the Galois group of $K : \mathbb{Q}$, where K is a splitting field over $\mathbb{Q}$ for t^4+t^2-6.
(b). Write down the subgroups of the Galois group.
(c). Write down the corresponding fixed fields.