PhD Algebra Examination

28 August, 1997

Answer seven out of the following ten problems. State clearly any results you need.

1. Let A be a commutative Noetherian ring with identity. Show that if A is Noetherian, then so is the power series ring A[[X]].

2. Suppose that A is a commutative ring with identity, B is an A-algebra integral over A, and $f : A \to K$ is a homomorphism from A to an algebraically closed field K. Show that f can be extended to a homomorphism $g : B \to K$.

3. Show that any automorphism of the field $\mathbb{R}$ of real numbers is the identity.

4. Let p be a prime, $n > 1$ an integer, and $G = GL(n, \mathbb{F}_p)$. Compute the order of G, and find a p-Sylow subgroup of G.

5. Let p be a prime.

(a) Show that if S_p is the symmetric group on p letters, and $H \subseteq S_p$ is a subgroup containing a p-cycle and a transposition, then $H = S_p$.

(b) Suppose $f(T) \in \mathbb{Q}[T]$ is irreducible of degree p. Show that if $f(T)$ has exactly two non-real roots, then the Galois group of the splitting field of $f(T)$ is S_p.

6. Let $M_n(F)$ denote the algebra of $n \times n$ matrices over the field F. Fix p with $1 < p < n$ and let R be the subalgebra consisting of those matrices in "block form"

$$\begin{pmatrix} A & B \\ 0 & C \end{pmatrix},$$

where A is a $p \times p$ matrix, and C is an $(n-p) \times (n-p)$ matrix.

(a) Describe the Jacobson radical of R.

Consider the vector space $V = F^n$ with its natural structure as a left module for R, given by matrix multiplication on the left.

(b) Show that V has an irreducible submodule W of dimension p and that V/W is also irreducible.

(c) Show that V is indecomposable, i.e. V is not the direct sum of two nonzero R-submodules.

7. If $(m, n) = 1$ then compute $(\mathbb{Z}/m\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z})$. Explain.

8. Suppose that A is a commutative ring with identity. If $A^m \simeq A^n$ as A-modules, show that $m = n$ (A^m denotes the direct sum of m copies of A as an A-module, and similarly for A^n).

9. Let A be a Noetherian ring and M a finitely generated A-module. Denote by Supp(M) the set of prime ideals p of A such that $M_p \neq 0$, where M_p is the localization of M at p. Show that if $M_p = 0$ for all p, then $M = 0$.

10. (a) Show that a direct sum of more than one copy of $\mathbb{Z}$ is not generated by a single element.

(b) Deduce that a direct sum of more than one copy of $\mathbb{Z}$ is not a free group.