PhD Algebra Examination

9 July, 1997

Answer seven out of the following ten problems. State clearly any results you need.

1. Let R be a commutative ring with identity, and S a multiplicative system in R. Show that the localization $S^{-1}R$ is a flat R-module.

2. Show that the field of complex numbers has uncountably many distinct automorphisms.

3. (a) Show that A_5 is simple.

(b) Use (a) to show that S_n is not solvable for $n \geq 5$.

4. If k is a field, show that the ring of power series $k[[T]]$ is a discrete valuation ring.

5. Suppose that p and q are distinct primes such that $q > p^2$ and $q \not\equiv 1 \pmod{p}$. Show that if G is a group of order $p^2 q$, then G is abelian.

6. Let R be a local noetherian (commutative) ring, and let M be a finitely generated R-module. Show that M is flat if and only if it is free.

7. Suppose that R is a ring with the property that every short exact sequence of unital R-modules splits. Show that every unital R-module is a direct sum of simple unital R-modules.

8. (a) Show that if k is a perfect field, then an irreducible polynomial in $k[T]$ has no multiple roots (Hint: show that if $f(T)$ is irreducible and has multiple roots, then the characteristic of k is $p > 0$, and $f(T) = g(T^p)$ for some $g \in k[T]$).

(b) Give an example of a (non-perfect) field k and an irreducible polynomial $f(T)$ with multiple roots.

9. Suppose R is a ring with identity. For each left R-module M, show that $\mathrm{Hom}_R(R, M)$ is naturally isomorphic to M (in the process, explain what "naturally isomorphic" means).

10. Classify up to isomorphism all semisimple rings of order 1584.