PhD Algebra Examination

22 May, 1996

Answer seven out of the following ten problems. State clearly any results you need.

1. Let R be an integral domain, with fraction field K. Recall that an R-submodule I of K is said to be *invertible* if $II^{-1} = R$, where $I^{-1} = \{x \in K \mid ix \in R \text{ for all } i \in I\}$. Show that if I is invertible, then it is finitely generated and projective.

2. (a) Let k be an algebraically closed field field and R a commutative, finitely generated k-algebra. Show that the Jacobson radical of R is the same as the nilradical. (Hint: Nullstellensatz).

(b) Give an example of a commutative ring with identity whose Jacobson radical is *not* equal to its nilradical.

3. Compute the Galois group of the extension $K/\mathbb{Q}$, where K is the splitting field over $\mathbb{Q}$ of the polynomial $X^4 - 3$.

4. Let K be a field, and R a simple, finite-dimensional K-algebra with identity (not necessarily commutative). Show that if the identity is the only (nonzero) idempotent element of R, then R is a division ring.

5. Let R be a discrete valuation ring with fraction field K. Show that R is integrally closed in K.

6. (a) Suppose G is a finite p-group and S is a finite set on which G acts. If S^G is the set of elements of S fixed by G, show that $|S| \equiv |S^G| \bmod p$.

(b) Suppose G is a p-group, k is a field of characteristic p, V is a finite-dimensional k-vector space, and $\rho : G \to GL(V)$ is a homomorphism. Show that there is a vector $v \in V$ such that $\rho(g)(v) = v$ for all $g \in G$. Hint: reduce to the case where k is finite, and apply (a).

7. Let k be a field and $G = GL(3, k)$. Describe the conjugacy classes of elements of order 3 in G if (a) $k = \mathbb{C}$, (b) $k = \mathbb{R}$, (c) $k = \mathbb{F}_3$.

8. Let R be a commutative ring. Define what it means for an R-module to be injective. Show that an R-module M is injective if and only if for every ideal $I \subseteq R$ and R-linear map $f : I \to M$, f extends to a map $R \to M$.

9. (a) Show that there exist solvable groups of arbitrarily large derived length.

(b) Show that if S is an infinite set, then there is no "free solvable group generated by S"; i.e. there is no solvable group F and map $\alpha : S \to F$ such that composition with α induces, for any solvable group G, a bijection between the set of group homomorphisms $F \to G$ and the set of maps $S \to G$.

10. State and prove the Hilbert Basis Theorem.