Ph. D. Exam in Algebra (5/24/95)

Time allowed: Four hours

Answer seven problems. You should indicate which problems you wish to have graded. Write your answers clearly in complete English sentences. You may quote results (within reason) as long as you state them clearly.

1. Let G be a finite group and let A be a subgroup of $\mathrm{Aut}(A)$ such that $G = [G, A]$. Let N be a normal subgroup of G such that $[N, A] = 1$. Prove that N is contained in the center of G.

2. (a) Give an example of fields $M \supset L \supset K$ such that M/L and L/K are normal extensions, but M/K is not normal. Justify your answer.
   (b) Let $L \supset K$ be fields such that $[L : K] = 8$. Prove that there are $\alpha, \beta, \gamma \in L$ such that $L = K(\alpha, \beta, \gamma)$.

3. Let A be a semi-simple commutative ring with 1. Prove that if A is Artinian then A is Noetherian. Give a counterexample to show that the converse is not true.

4. (a) Give an example of a free module F over a ring with 1 which has a submodule M which is not free.
   (b) Let A be a commutative ring with 1 and let P and Q be projective A-modules. Prove that $P \otimes Q$ is projective.

5. Let A be a commutative ring with 1 and let $S \not\ni 0$ be a multiplicative subset of A. Let P be a maximal element in the set $\{I : I \text{ is an ideal of } A \text{ and } I \cap S = \emptyset\}$. Prove that P is a prime ideal.

6. Let A be a commutative ring.
   (a) Prove that if I_1 and I_2 are ideals in A and P is a prime ideal in A such that $I_1 \cap I_2 = P$ then $I_1 = P$ or $I_2 = P$.
   (b) Is the above statement still true if the word "prime" is replaced by "primary"? Justify your answer.

7. Let n and m be positive integers. Find r such that $\mathbf{Z}_n \otimes_{\mathbf{Z}} \mathbf{Z}_m \cong \mathbf{Z}_r$. Justify your answer.

8. Let R be a commutative Noetherian ring with 1. Prove that $R[[X]]$ is Noetherian.

9. Let R be a commutative ring with 1 and let $S \subset R$ be a multiplicative set which contains no zero divisors. Prove that the prime ideals of the ring $S^{-1}R$ are in one-to-one correspondence with the prime ideals P of R such that $P \cap S = \emptyset$.

10. Let G be an abelian group which is injective. Prove that G is divisible.

11. Let $f(X) = 2X^5 - 10X + 5 \in \mathbf{Q}[X]$ and let L be a splitting field for f. Compute the Galois group $\mathrm{Gal}(L/\mathbf{Q})$.