PhD Algebra Examination, September, 1994

WORK 7 OUT OF THE FOLLOWING 11 EXERCISES

1. (a) Prove that any finite p-group is nilpotent.
   (b) If G is a finite group which is a direct product of its Sylow subgroups, show that G is nilpotent.

2. (a) Define *solvable group*.
   (b) Prove that the homomorphic image of a solvable group is solvable.
   (c) Prove that a free group is solvable if and only if it is the free group on at most one generator.

3. State and prove the Jacobson Density Theorem. (You may assume the following lemma: Let R be a ring with identity, and suppose that S is a simple R-module. View S as a vector space over the division ring $D = \mathrm{Hom}(S,S)$. If V is any finite dimensional D-subspace of S, and $a \in S\backslash V$, then there is an element r in R so that $rV = 0$ but $ra \neq 0$.)

4. Suppose that A is a commutative ring with identity. If P and Q are projective A-modules, prove that $P \otimes_A Q$ is also projective.

5. Apply the Wedderburn-Artin Theorem to characterize the left Artinian rings with identity for which $r^3 = r$, for each $r \in R$.

6. Let A be a commutative ring with identity, and suppose that M is an A-module, and I is an ideal of A. Prove that $(A/I) \otimes M = M/IM$, where IM is the submodule generated by all elements of the form xb, with $x \in I$, $b \in M$.

7. Let k be a field; $A = k[T_1, T_2, \ldots, T_n]$ stands for the polynomial ring in n indeterminates.
   (a) Define: the *affine variety* associated with an ideal I of A.
   (b) The affine varieties are the closed subsets of a topology on k^n; accepting this, prove that the closed subsets of k^n satisfy the descending chain condition.

8. Suppose that A is a subring of the commutative ring B with 1. If B is integral over A, prove that every homomorphism f of A into the algebraically closed field L admits an extension to a homomorphism $g: B \longrightarrow L$.

9. Prove that the lattice of all ideals of a Dedekind domain is distributive. (Hint: Localize!)

10. (a) Define: splitting field of a polynomial.

    (b) Assuming existence and uniqueness of splitting fields, up to isomorphism over the base field, prove this:

Let E be a finite extension of F. Then E is a splitting field for some polynomial if and only if every irreducible polynomial over F, having a root in E, factors completely over E.

11. Let p be a prime number.

    (a) Prove that if a subgroup H of S_p, the symmetric group on p letters, contains a p-cycle and a transposition, then $H = S_p$.

    (b) If $p(T)$ is an irreducible polynomial over Q, of degree p, having exactly two non-real roots, then show that the Galois group of the splitting field of $p(T)$ is S_p.