Suggested PhD Algebra Examination, May, 1994

WORK 7 OUT OF THE FOLLOWING 11 EXERCISES

1. Suppose that F is a free group on the alphabet X, and that Y is a subset of X. Let H be the least normal subgroup of F containing Y. Prove that F/H is a free group. (Hint: Show it's free on the alphabet $X \setminus Y$.)

2. Suppose that G is a finite group.
   (a) Prove that if G is nilpotent, and H is any proper subgroup, then H is a proper subgroup of its normalizer.
   (b) Use (a) to prove that if G is nilpotent then it is isomorphic to a finite direct product of p-groups.

3. Suppose that R is a principal ideal domain. Prove that any submodule of a free R-module is free.

4. Prove that there are (up to ring isomorphism) only 12 semisimple left Artinian rings of order 1008, of which only two are not commutative.

5. Suppose that A is a commutative ring with identity. Suppose that $F(m)$ and $F(n)$ are free modules on m and n generators, respectively. Prove that if $F(m) \cong F(n)$, then $m = n$.

6. Suppose that R is a ring with identity. Prove that

$$\mathrm{Hom}_{\mathbb{Z}}(B, \times_{i\in I} G_i) = \times_{i\in I} \mathrm{Hom}_{\mathbb{Z}}(B, G_i),$$

   as right R-modules, for all left R-modules B and all abelian groups G_i $(i \in I)$. You may use resources from category theory; if so, outline your argument so that it is clear which theorems you are appealing to.

7. Let A be a commutative ring with identity. For each multiplicative system S of A, prove that $S^{-1}A$ is a flat A-module.

8. State and prove the Hilbert Nullstellensatz. (You may use the following: if K is an algebraically closed field, A a finitely generated K-algebra, and M is a maximal ideal of A, then A/M is isomorphic to K.)

9. Among the following integral domains, decide which ones are Dedekind domains, and give a brief explanation. Quoting a relevant theorem will do; likewise, a well-illustrated example.

   (a) $Z[T]$, the polynomial ring over the integers, in one variable.

   (b) $Z[\sqrt{(-5)}] = \{ a + b\sqrt{(-5)} : a, b \in Z \}$.

   (c) The ring $k[[T]]$ of all formal power series in one variable, over the field k.

10. Suppose that E is a finite Galois extension of the field F. If $Gal(E/F)$ has order pq, where $p < q$ are primes, such that p does not divide $q - 1$, prove that E has two subfields E_p and E_q, which are stable under the action of $Gal(E/F)$, such that $E_p \cap E_q = F$, E_p and E_q generate E, and $Gal(E_p/F)$ (resp. $Gal(E_q/F)$) is cyclic of order p (resp. q).

11. (a) Define: algebraic closure.

    (b) Prove that every field has an algebraic closure, and argue that if E is an algebraic closure of F, then $|E|$ is countable if F is finite, while $|E| = |F|$, otherwise.