PhD ALGEBRA EXAMINATION
May 20, 1993
6:00 p.m. - 10:00 p.m.

Do seven of the following questions (the exam committee will NOT select your best seven answers if you answer more than seven). You may quote standard results (within reason) as long as you make it clear that are doing so and you state them clearly.

---

1. Define what is meant by a free group. Let F be a free group on generators X and let $Y \subset X$. Prove that if H is the smallest normal subgroup of F generated by Y then F/H is a free group.

2. Define what is meant by a nilpotent group. Show that if G is a finite group then G is nilpotent if and only if every Sylow subgroup of G is a normal subgroup of G.

3. State and prove Hilbert's Theorem 90 about the elements of norm 1 in certain Galois extensions.

4. Let F be a field of characteristic zero and let M be a finite dimensional vector space over F. Let b be a symmetric bilinear form on M. Prove that M has a basis $e_1, \ldots, e_n$ such that $b(e_i, e_j) = 0$ whenever $i \neq j$.

5. Prove: let S be an integral extension ring of the domain R, and let P be a prime ideal of R. Then there is a prime ideal Q of S such that $Q \cap R = P$.

6. Let F be an algebraically closed field extension of a field K.
(i) Prove that there is a one-to-one inclusion reversing correspondence between the set of affine K-varieties in F^n and the set of radical ideals of $K[x_1, \ldots, x_n]$.
(ii) If $V_1 \supset V_2 \supset \ldots$ is a descending chain of K-varieties in F^n, then $V_m = V_{m+1} = \ldots$ for some m.

7. Let F be a field and let A and B be simple F-algebras. Assume that A is central (that is that $\mathrm{Z}(A) = F.1$). Prove that $A \otimes B$ is a simple F-algebra.

8. Prove Wedderburn's Theorem that every finite division ring is a field.

9. Let A and B be abelian groups. Prove
(i) For each $m > 0$, $A \otimes \mathbb{Z}_m \simeq A/mA$.
(ii) $\mathbb{Z}_m \otimes \mathbb{Z}_n \simeq \mathbb{Z}_e$, where $e = (m, n)$.
(iii) Describe $A \otimes B$, when A and B are finitely generated.

10. Prove that no root of the polynomial $x^5 - 6x - 2$ can be obtained from rational numbers by using only the operations of addition, subtraction, multiplication, division and extraction of n-th roots for various n.

11. Let R be an associative ring.

(a) What does it mean for the R-module P to be:

(i) free?

(ii) projective?

(b) Are free modules always projective? Are projective modules always free? Justify your answers.

(c) Give an example of a noncommutative ring for which the projective modules and the free modules are the same.