Ph.D. Algebra
September 10, 1990

Time allowed: 4 hours

Read through the questions carefully before you begin. Your six best solutions will count. Theorems from the course may be quoted (within reason) as long as you give a clear indication and state the result in full.

1. Let P be a nontrivial, finite P-group, and Q a subgroup. Consider the action of P on the set P/Q of left cosets of Q by left multiplication:

$$P \times P/Q \longrightarrow P/Q$$
$$(P, xQ) \longrightarrow (px)Q.$$

Q acts on P/Q by restricion a) Show that xQ is a fixed point of Q if and only if $x \in N_P(Q) = \{p \in P \mid pQp^{-1} = Q\}$.

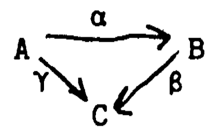
b) Deduce that if Q is a proper subgroup of P then Q is also a proper subgroup of $N_p(Q)$.

c) Hence show that every maximal subgroup of P is normal of index p.

2. Let A be a Dedekind domain with field of fractions K. Show that every ideal of A can be generated by two elements. (Hint: use the Chinese remainder theorem.)

3. Prove that any simple group of order 60 is isomorphic to A_5. (you need not show that A_5 is simple).

4. Let A, B and C be rings (with 1) and suppose we have a commutative diagram of ring homomorphisms:



a) Define the structure of an (A,B) bimodule on B, ie make B into a left A-module and a right B-module such that $(a.x)G = a(x.b)$ for $a \in A$, $x, b \in B$.

b) Show that for any right A-module M, we have a natural isomorphism

$$M \otimes_A C \cong (M \otimes_A B) \otimes_B C.$$

(Remember what 'natural' means in terms of the functors.

$$- \otimes_A C \text{ and } (- \otimes_A B) \otimes_B C.)$$

5. Let $\{\sigma_i\}_{i=1}^{n}$ be a set of n distinct automorphisms of a field k. Show that if for $a_1, .. a_n \in k$

$$\Sigma a_i \sigma_i(u) = 0 \qquad \text{for all } u \in k$$

then $a_i = 0$ for all i.
(Hint show that there is no 'shortest' relation with $a_i \neq 0$).

6. Give examples of the following (with proof).

a) An irreducible cubic polynomial with rational coefficients whose Galois group has order 3.

b) Two normal extensions $K \subset L$, $L \subset M$ such that extension $K \subset M$ is not normal.

7. Let R be a ring and M simple (irreducible) R-module.

a) Prove Schur's Lemma: every nonzero R-module map from M to itself is an isomorphism.

b) Give an example to show that the converse of Schur's Lemma is false.

8. Let $\mathbb{Z}_p$ denote the p-adic integers and for $n \in N$

$\varepsilon_n: \mathbb{Z}_p \to \mathbb{Z}/p^n\mathbb{Z}$ the map defining the structure of $\mathbb{Z}_p$ as

$\varprojlim_n \mathbb{Z}/p^n\mathbb{Z}$. Let $U_n = 1 + p^n\mathbb{Z}_p$.

a) Show that U_n is a subgroup of the group U of units of $\mathbb{Z}_p$.

b) Show that U_n is the kernel of $\varepsilon_{n/U}: U \to (\mathbb{Z}/p^n\mathbb{Z})^*$ and that $U/U_1 \cong F_p^*$ is cyclic of order p-1.

c) For $n \geq 1$ show that the map

$1 + p^n x \dashrightarrow x$ modulo p defines an isomorphism

$U_n/U_{n+1} \xrightarrow{\sim} Z/_pZ$.

d) Deduce that U_1/U_n is a p-group and that U/U_n has a unique subgroup of order p-1.

e) Prove that U has a subgroup of order p-1.
(Hint: U: $\varprojlim U/U_n$)

9. Let D be a division ring and M a D-module. A subring R of $End_D(M)$ is said to be <u>dense</u> if for any natural number n and $v_1, \ldots, v_n \in M$ which are linearly independent over D, and any n elements $W_1, \ldots, W_n$ of M, there exists $r \in R$ such that $W_i = r(v_i)$ i = 1, ..., n.

a) Show that if $\dim_D M$ is finite then a dense subring of $End_D M$ must be equal to $End_D M$.

b) Show that if M is infinite-dimensional over D and R is a dense subring of $End_D M$, then for every natural number m, R has a subring S_m which maps homomorphically onto $Mat_m(D)$.

10. a) State Hilbert's Nullstellensatz, giving the main definitions involved.

b) Let k be an algebraically closed field. Show that every algebraic subset of k^n can be decomposed uniquely as a finite union of (irreducible) varieties.

c) Give an example to show that in b) the union may not be disjoint.