# Ph.D. Level Examination in Algebra
## Summer 1989

**Do 7 out of the following questions:.**

1. (a) Define: Solvable group.

(b) State the Burnside's theorem for solvability of a finite group.

(c) Let $H$ be a normal subgroup of a group $G$. Prove that $C_G(H)$ is a normal subgroup of $G$.

(d) Let $G$ be a finite group of order 12. Prove that $G$ has a non-trivial normal 2-subgroup (i.e., there is a subgroup $T$ of $G$ such that $1 \neq T$, and $T$ is a 2-subgroup , and $T$ is normal in $G$).

2. Two permutation representations $\phi_i : G \rightarrow \Sigma(X_i)$, (the symmetric group on $X_i$), are equivalent if there exisits $\alpha : X_1 \rightarrow X_2$ such that for all $g \in G$, $x \in X_1$, we have $x(g\phi_1)\alpha = x(\alpha)(g\phi_2)$.

Prove or disprove that the following two permutation representations of $S_4$ of degree 6 are equivalent:

(a) on the right coset space of $< (12)(34), (13)(24) >$.

(b) on the right coset space of $< (12), (34) >$.

3. Let $F_n$ be the free group on n generators $x_1, ..., x_n$. Let $F_n'$ denotes the commutator subgroup. Prove that $F_n/F_n'$ is the free abelian group on $\{x_1, ..., x_n\}$.

4. In each of the following groups decide whether it is a semi-direct product of two of its proper subgroups: $D_4$ (the dihedral group of order 8); $Q_8$ (the quaternion group of order 8). Justify your answer.

5. Let $A$ be a commutative ring with indentity. Suppose that for each prime ideal $P$, the local ring $A_P$ has no nilpotent element $\neq 0$. Show that $A$ has no nilpotent element $\neq 0$. If each $A_P$ is an integral domain, is $A$ necessarily an integral domain? Justify your answer.

6. Let $A$ be a commutative ring with identity, $a$ an ideal, $M$ an $A$-module. (a) Prove that $A \otimes_A M \cong M$ as $A$-modules.

(b) Prove that $(A/a) \otimes_A M$ ia isomorphic to $M/aM$.

7. Let $A \subseteq B$ be integral domains, $B$ integral over $A$. Prove that $B$ is a field if and only if $A$ is a field.

8. Find a minimal primary decomposition for the following ideals. Also give the prime ideals to which the primary ideals belong.

(a) (36) as an ideal in $Z$.

(b) $(x^2, xy)$ as an ideal in $C[x, y]$.

9. State the following theorems.

(a) Nakayama's lemma.

(b) Hilbert basis theorem.

(c) Jacobson's Density theorem.

(d) Maschke's theorem.

(e) Wedderburn's theorem for finite dimensional simple algebras (or Wedderburn-Artin's theorem for simple Artinian rings).