Ph.D. Level Examination in Algebra

Fall 1988

Do <u>seven</u> problems, at least <u>two</u> from each section.

I. Group Theory.

1. Let G be the group of all 3×3 non-singular matrices of determinant 1 with coefficient in the field of 3 elements.

   (a) Find $|G|$.

   (b) Find a Sylow 3-subgroup of G.

   (c) Compute $|Syl_3(G)|$, where $Syl_3(G)$ is the set of all Sylow 3-subgroups of G.

2. Let G be a finite group acting transitively on a set X, $x \in X$, and P be a Sylow p-subgroup of G_x for some prime p. Consider the set $Fix(P) = \{y \in X | y^P = y\}$, the set of fixed points of P.

   (a) Prove that $N_G(P)$ acts on Fix(P), i.e., maps Fix(P) to Fix(P).

   (b) Prove that $N_G(P)$ acts transitively on Fix(P).

3. Two permutation representations $G \xrightarrow{\pi_i} \Sigma(X_i)$, (the symmetric group on X_i are equivalent if there exists $\alpha: X_1 \to X_2$ such that for all $g \in G$, $x \in X_1$, we have $x(g\pi_1)\alpha = (x\alpha)(g\pi_2)$.

   Prove that the following two permutations of S_4 of degree 12 are not equivalent:

   (a) on the right coset space of $\langle(12)\rangle$.

   (b) on the right coset space of $\langle(12)(34)\rangle$.

4. Let G be a finite group and $A \leq Aut(G)$ such that $G = [G,A]$. Suppose $N \trianglelefteq G$ satisfying $[N,A] = 1$. Use the 3-subgroup lemma to prove $N \leq Z(G)$, the center of G.

   (Recall the 3-subgroups lemma: For three subgroups X, Y, Z of a group M if $[X,Y,Z] = 1$ and $[Y,Z,X] = 1$, then $[Z,X,Y] = 1$.)

II. Homological algebra.

1. Let R be a ring and consider the following commutative diagram of R-modules and R-module homomorphisms such that each row is a short exact sequence.

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \\
0 & \longrightarrow & A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \longrightarrow & 0
\end{array}
$$

Prove that if α and γ are isomorphisms then β is also an isomorphism.

2. Prove the Z-module isomorphisms:

(a) $Z/mZ \otimes Z/nZ \cong Z/cZ$

(b) $\mathrm{Hom}(Z/mZ, Z/nZ) \cong Z/cZ$

where $c = (m,n)$ is the greatest common divisor.

3. A module P over a ring R is said to be projective if given any diagram of R-module homomorphisms with g surjective

$$
\begin{array}{ccc}
 & & P \\
 & & \downarrow{\scriptstyle f} \\
A & \xrightarrow[g]{} & B.
\end{array}
$$

there exists an R-module homomorphism h: P ——> A such that the following diagram is commutative

$$
\begin{array}{ccc}
 & & P \\
 & {\scriptstyle h}\swarrow & \downarrow{\scriptstyle f} \\
A & \xrightarrow[g]{} & B
\end{array}
$$

Prove that the following conditions on a ring R with identity are equivalent:

(a) Every R-module is projective.

(b) Every short exact sequence of R-modules splits.

III. Commutative algebra.

1. A non-empty subset S of a ring R with identity is called multiplicative if $ab \in S$ whenever $a,b \in S$.

   (a) If I is a proper ideal of R disjoint from S, then prove that there exists an ideal P maximal with respect to containing I and being disjoint from S. Furthermore, show that P is a prime ideal.

   (b) If P is a prime ideal of ring R, define the localization of R at P and prove that the localization has a unique maximal ideal.

2. Find a reduced primary decomposition for the following ideals. Also give the prime ideals to which the primary ideals belong.

   (a) (24) as an ideal in Z.

   (b) $(2x, x^2)$ as an ideal in Z[x].

   (c) What theorems allow us to conclude that every ideal in Z[x] has a primary decomposition.

3. Let R be a unique factorization domain and K its field of fractions.

   (a) Show that if $a \in K$ is integral over R then $a \in R$. (In other words, R is integrally closed.)

   (b) True or false: If S is a subring of K containing R, then S is also a unique factorization domain. If true, prove your answer. If false give examples of R, K and S that show it not to be true.

4. Determine, up to isomorphism, all semisimple rings of order $1008 = 2^5 \cdot 3^2 \cdot 7$.