Ph. D. Algebra Exam **August 18, 2016**

Answer seven problems. You should indicate which problems you wish to have graded. Write your answers clearly in complete English sentences. You may quote results (within reason) as long as you state them clearly.

1. Let L be a splitting field over $\mathbb{Q}$ for the polynomial $f(X) = X^3 - 7$.
   (a) Determine the Galois group $\mathrm{Gal}(L/\mathbb{Q})$.
   (b) For each subgroup H of $\mathrm{Gal}(L/\mathbb{Q})$ determine the fixed field L^H of H.

2. Let p be prime and let $n \geq 1$.
   (a) Prove that there is a field $\mathbb{F}_{p^n}$ with p^n elements.
   (b) Prove that if K is a field with p^n elements then $K \cong \mathbb{F}_{p^n}$.

3. (a) Let $\mathcal{C}$ be a category. Define what it means for a $\mathcal{C}$-morphism $f : X \to Y$ to be a monomorphism.
   (b) Give the definition of a concrete category.
   (c) Let $\mathcal{C}$ be a concrete category and let $f : X \to Y$ be a $\mathcal{C}$-morphism whose underlying set map is one-to-one. Prove that f is a monomorphism.

4. Prove that the group G defined by the generators and relations
$$G = \langle x, y \mid xyx = y \rangle$$
is infinite and nonabelian.

5. Let R be a commutative ring with 1 and let I, J be ideals in R. Prove that there is an isomorphism of R-modules
$$(R/I) \otimes_R (R/J) \cong R/(I + J).$$

6. Let R be a commutative ring with 1.
   (a) Define what it means for a left R-module to be flat.
   (b) Prove that if F is a free left R-module then F is flat.

7. Let $R \subset S$ be integral domains such that S is integral over R. Prove that R is a field if and only if S is a field.

8. Let R be a commutative ring with 1.

   (a) Define the Jacobson radical $J(R)$ of R.

   (b) Let M be a finitely generated R-module and let I be an ideal of R which is contained in $J(R)$. Prove that if $IM = M$ then $M = \{0\}$.

9. Determine which of the following rings are Dedekind domains, with some explanation in each case.

   (a) $\mathbb{F}_7[X]$

   (b) $\mathbb{F}_7[X, Y]$

   (c) $\mathbb{Z}[\sqrt{3}]$

   (d) $\mathbb{Z}[\sqrt{5}]$

10. Let G be a finite group and let K be a field of characteristic 0. Let V be a $K[G]$-module which is a finite-dimensional vector space over K and let W be a $K[G]$-submodule of V. Prove there is a $K[G]$-submodule W' of V such that $V = W \oplus W'$.

11. Determine the number of isomorphism classes of semisimple rings with 4000 elements. You don't have to list the rings, but you do need to justify your computations.