Ph. D. Algebra Exam **August 16th, 2018**

Answer **seven** problems. You should indicate which problems you wish to have graded. Write your answers clearly in complete English sentences. You may quote results (within reason) as long as you state them clearly.

1. (10 points) Determine the Galois group of the polynomial
$$f = x^5 - 2 \in \mathbb{Q}[x].$$

2. (10 points) Let F be a field of characteristic zero and E a finite extension field of F. Prove that E is a simple extension of F, i.e. there exists $\alpha \in E$ such that $E = F(\alpha)$.

3. (10 points) Let R be a ring with 1. Prove the equivalence of the following conditions.
   (a) Every unital left R-module is projective.
   (b) Every short exact sequence of unital left R-modules splits.
   (c) Every unital left R-module is injective.

4. (10 points) Prove that the additive group of the field of rational numbers is not a projective $\mathbb{Z}$-module.

5. Let R and S be rings, A a right R-module, B an (R, S)-bimodule and C a left S-module.
   (a) (2 points) Explain how $(B \otimes_S C)$ is a left R-module and $(A \otimes_R B)$ is a right S-module.
   (b) (8 points) Prove that we have an isomorphism (of abelian groups)
$$A \otimes_R (B \otimes_S C) \cong (A \otimes_R B) \otimes_S C.$$

6. (10 points) Let R be an integral domain and for each maximal ideal M, regard the localization R_M as a subring of the field of fractions F of R. Show that $\cap_M R_M = R$. (Hint: for $x \in F$, consider the ideal $D = \{a \in R \mid ax \in R\}$. If $x \in R_M$ what does it say about D?)

7. (10 points) Let R be a Dedekind domain and I a nonzero ideal of R.
   (a) Prove that R/I is Artinian.
   (b) Prove that if R is itself Artinian, then it is a field.

8. Let $F[[X]]$ denote the ring of formal power series over the field F.
   (a) (5 points) Prove that $a_0 + a_1X + a_2X^2 \cdots \in F[[X]]$ is a unit if and only if $a_0 \neq 0$.

(b) (5 points) Prove that $F[[X]]$ is a discrete valuation ring.

9. (10 points) Let R and S be commutative rings with 1. Prove the *Lying-Over Theorem*: Let S be an integral extension of R and P a prime ideal of R. Then there exists a prime ideal Q of S such that $Q \cap R = P$.

10. (10 points) Prove that the ring of $n \times n$ matrices with entries in a field F is a simple ring when $n \geq 1$. Describe explicitly one of its minimal left ideals.

11. Let $\mathcal{C}$ be a category and let Λ be a set. Suppose that for each $\lambda \in \Lambda$ we are given an object X_λ in $\mathcal{C}$.

    (a) (3 points) Give the definition of a *product* of the collection of objects $\{X_\lambda\}_{\lambda\in\Lambda}$.

    (b) (2 points) Prove that any two products of the $\{X_\lambda\}_{\lambda\in\Lambda}$ are isomorphic.

    (c) (3 points) Prove that in the category of Groups a product exists for any collection of objects indexed by a set Λ.

    (d) (2 points) Prove that if, in (c), the groups in the collection happen to be abelian, then a product as in (c) is also a product in the category of Abelian groups.