Topology Ph.D. Exam
May 2019

Work the following problems and show all your work. Support all statements to the best of your ability. Work each problem on a separate sheet of paper.

1. Prove that the topologist's sine curve $T = \{(x, \sin\frac{1}{x}) \mid x \in (0,1]\} \cup \{(0,y) \mid y \in [0,1]\}$ is not path connected.
2. Let X be a connected completely regular topological space having more than one point. Can X be countable?
3. Show that the 2-sphere S^2 is not a retract of the real projective plane $\mathbb{RP}^2$, as well as that $\mathbb{RP}^2$ is not a retract of S^2.
4. Consider the Klein Bottle, K. Give its cellular chain complex and use it to calculate the homology groups of K with integer coefficients.
5. Let $n > m$. Show that there is no map $f : \mathbb{CP}^n \to \mathbb{CP}^m$ inducing a nontrivial map $f^* : H^2(\mathbb{CP}^m; \mathbb{Z}) \to H^2(\mathbb{CP}^n; \mathbb{Z})$.

Answer the following with complete definitions or statements or short proofs.

6. Draw a picture of the universal cover of the 2-sphere with a segment joining the north and south poles.
7. Compute the Euler characteristic $\chi(\mathbb{CP}^2 \times \mathbb{RP}^2 \times S^4)$.
8. Give a definition of the compact-open topology for the space of functions.
9. Construct a map $f : T^2 \to S^2$ of degree 3 where $T^2 = S^1 \times S^1$ is a torus.
10. Does every function $f : \mathbb{N} \to S^1$ admit a continuous extension $\overline{f} : \beta\mathbb{N} \to S^1$ to the Stone–Čech compactification?
11. State the exact homology sequence for a pair of spaces.
12. What can you say about the k-th cohomology group of a closed orientable manifold of dimension n for (a) $k = n$? (b) $k = n - 1$?
13. State the Universal Coefficient Theorem for Cohomology.
14. Show that two closed orientable surfaces of different genus are not homeomorphic.
15. State the Tietze Extension Theorem.