Ph.D. Examination – Topology
May 2017

Work the following problems and show all work. Support all statements to the best of your ability. Work each problem on a separate sheet of paper.

1. Prove that a compact Hausdorff space is regular.

2. Prove that if $n > 0$ every map $S^n \to S^n$ is homotopic to a map with a fixed point.

3. The oriented surface M_g of genus g, embedded in $\mathbb{R}^3$ in the standard way, bounds a compact region R. Two copies of R, glued together by the identity map between their boundary surfaces M_g, form a closed 3-manifold X. Compute the homology groups of X using the Mayer–Vietoris sequence.

4. Let $f : S^1 \to \mathbb{R}$ be a continuous map. Prove that there exists a point $x \in S^1$ with $f(x) = f(-x)$. (Note: do not simply state the Borsuk–Ulam Theorem; give a direct proof.)

5. Let M be a closed simply-connected orientable 3-manifold. Compute the integral homology and cohomology of M. What can you say about $\pi_i(M)$, $i \leq 3$?

Answer the following with complete definitions, statements, or short proofs.

6. Prove that for a finite CW-complex X, $H^1(X; \mathbb{Z})$ is torsion-free.

7. Compute $\chi(\mathbb{C}P^3 \times \mathbb{R}P^2 \times S^2)$

8. Give an example of a space that is path-connected but not locally path-connected.

9. State the Urysohn Lemma.

10. Prove that if $m \neq n$, then $\mathbb{R}^m$ is not homeomorphic to $\mathbb{R}^n$.

11. Does the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \arctan x$ admit a continuous extension $\overline{f} : \beta\mathbb{R} \to \mathbb{R}$ to the Stone–Čech compactification? What about the function $g(x) = e^x$?

12. State the Lefschetz Fixed Point Theorem.

13. Describe all the connected covering spaces $E \to S^1$.

14. Does the following exact sequence of abelian groups necessarily split? Prove or give a counterexample.

$$0 \to \mathbb{Z} \to A \to B \to 0$$

15. Compute the integral homology of the space $\mathbb{R}P^4 \times \mathbb{C}P^2$.