Applied Mathematics PhD Qualifier Exam – January 3, 2018
Do **8** (eight) problems

1. Let $\mathbf{Q}$ denote an $n$ by $n$ symmetric matrix and let $\mathbf{d}_k$, $1 \leq k \leq n$, denote a collection of directions that are $\mathbf{Q}$-conjugate.

   (a) Suppose the $\mathbf{Q}$ is positive definite. Show that the directions $\mathbf{d}_k$, $1 \leq k \leq n$, are linearly independent, and express the solution of the linear system $\mathbf{Qx} = \mathbf{b}$ in terms of the conjugate directions.

   (b) If $\mathbf{d}_k^\mathsf{T}\mathbf{Q}\mathbf{d}_k > 0$ for each $k$, then show that $\mathbf{Q}$ is positive definite.

2. Consider the following variational problem:

$$\min\left\{\int_0^1 \frac{1}{2}u'(x)^2 + e^{u(x)}dx : u \in H^1([0,1]), \quad u(0) = u(1) = 0\right\}$$

   (a) What is the first-order necessary optimality condition (Euler equation) for this variational problem?

   (b) Consider a uniform mesh $x_k = kh$ where $h = 1/N$; let $u_k$ denote an approximation to $u(x_k)$. Of course, $u_0 = u_N = 0$. Give a finite difference approximation in terms $u_1, \ldots, u_{N-1}$ to the solution of the Euler equation.

   (c) Let $\mathbf{F}(\mathbf{u})$ denote the finite difference system where $\mathbf{u} = (u_1, u_2, \ldots, u_{N-1}) \in \mathbb{R}^{N-1}$, and let $\mathbf{u}^*$ denote the vector formed by evaluating the solution of the Euler equation at the mesh points $x_1, x_2, \ldots, x_{N-1}$. Obtain a bound for the components of $\mathbf{F}(\mathbf{u}^*)$ in terms of the mesh spacing $h$.

3. Consider the following variational problem where $f$ is a given smooth function:

$$\min\left\{\Phi(u) := \int_0^1 \frac{1}{2}u'(x)^2 + f(x)u(x)\, dx : u \in H^1([0,1]), \quad u(0) = u(1) = 0\right\}.$$

   Let $\mathcal{S}^h$ denote the piecewise linear finite element space defined on a uniform mesh with $v^h(0) = v^h(1) = 0$ for all $v^h \in \mathcal{S}^h$. Obtain a bound for the error in the finite element approximation $u^h$ obtained by minimizing $\Phi$ over $\mathcal{S}^h$.

4. Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ is twice continuously differentiable, $\mathbf{x}^* \in \mathbb{R}^n$, $\nabla f(\mathbf{x}^*) = 0$, and $\nabla^2 f(\mathbf{x}^*)$ is positive definite. Show that $\mathbf{x}^*$ is a strict local minimizer of $f$.

5. Suppose that a quadratic $\Phi(\mathbf{x}) = \frac{1}{2}\mathbf{x}^\mathsf{T}\mathbf{Q}\mathbf{x} - \mathbf{b}^\mathsf{T}\mathbf{x}$ is minimized by steepest descent: $\mathbf{x}_{k+1} = \mathbf{x}_k - s_k\mathbf{g}_k$, where $\mathbf{g}_k = \nabla\Phi(\mathbf{x}_k)$ and $s_k$ is the stepsize.

   (a) Derive the formula for the Cauchy step.

   (b) Derive the formula for the BB step.

6. Consider the system of differential equations $\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t))$, $\mathbf{x}(0) = \mathbf{x}_0$, where $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ and $\mathbf{f}$ satisfies the global Lipschitz condition

$$\|\mathbf{f}(\mathbf{x}_1, \mathbf{u}_1) - \mathbf{f}(\mathbf{x}_2, \mathbf{u}_2)\| \leq L(\|\mathbf{x}_1 - \mathbf{x}_2\| + \|\mathbf{u}_1 - \mathbf{u}_2\|).$$

Show that the differential equation has a global solution and on any interval $[0, T]$, the following bound holds for all $(\mathbf{u}_1, \mathbf{u}_2) \in \mathcal{L}^\infty([0, T])$:

$$\|\mathbf{x}_1 - \mathbf{x}_2\|_{\mathcal{L}^\infty} \leq e^{LT} \|\mathbf{u}_1 - \mathbf{u}_2\|_{\mathcal{L}^1},$$

where $\mathbf{x}_i$ is the solution of the differential equation associated with $\mathbf{u}_i$, $i = 1, 2$.

7. Suppose that $P$ and $Q$ are continuous on $[0, 1]$ and the following functional is nonnegative over all $h \in \mathcal{H}_0^1([0, 1])$:

$$\Omega(h) = \int_0^1 P(x)h'(x)^2 + Q(x)h(x)^2 \, dx.$$

Show that $P \geq 0$ on $[0, 1]$.

8. Suppose that $u : [0, 1] \to \mathbb{R}$ is twice continuously differentiable, and let $u^I$ be the continuous, piecewise linear interpolant of $u$. Prove the following bound:

$$\|u - u^I\|_{\mathcal{L}^\infty} \leq \frac{h^2}{8} \|u''\|_{\mathcal{L}^\infty}.$$

9. Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ is a convex function.

   (a) If $f$ is continuously differentiable, then show that for all $\mathbf{x}$ and $\mathbf{y} \in \mathbb{R}^n$, we have

   $$f(\mathbf{y}) \geq f(\mathbf{x}) + \nabla f(\mathbf{x})(\mathbf{y} - \mathbf{x}).$$

   (b) If $f$ is twice continuously differentiable, then show that the Hessian $\nabla^2 f(\mathbf{x})$ is positive semidefinite for all $\mathbf{x} \in \mathbb{R}^n$.

10. Suppose that $\mathbf{\Phi} : \mathbb{R}^n \to \mathbb{R}^n$ and $\mathbf{x}^* \in \mathbb{R}^n$ is a fixed point of $\mathbf{\Phi}$.

   (a) If $\mathbf{\Phi}$ is a contraction mapping with contraction constant $\lambda$, then show that the iteration $\mathbf{x}_{k+1} = \mathbf{\Phi}(\mathbf{x}_k)$ converges to $\mathbf{x}^*$ from any starting guess $\mathbf{x}_0$, and

   $$\|\mathbf{x}_k - \mathbf{x}^*\| \leq \lambda^k \|\mathbf{x}_0 - \mathbf{x}^*\|.$$

   (b) Suppose that $\mathbf{\Phi}$ is continuously differentiable on a convex set $\mathcal{K} \subset \mathbb{R}^n$. Let $\mu$ denote the supremum of the singular values of $\nabla \mathbf{\Phi}$ over $\mathcal{K}$. Show that

   $$\|\mathbf{\Phi}(\mathbf{x}) - \mathbf{\Phi}(\mathbf{y})\| \leq \mu \|\mathbf{x} - \mathbf{y}\|$$

   for all $\mathbf{x}$ and $\mathbf{y} \in \mathcal{K}$.