Numerical Analysis Exam – August, 2017

Do all **5** (five) problems

1. Let $f(x) = e^x$.

   (a) Find the linear Taylor polynomial $T_1(x)$ of $f(x)$ expanded about $x_0 = 1/2$ and give an estimate for the maximum of $|T_1(x) - f(x)|$ on the interval $[0, 1]$.

   (b) Find the linear minimax approximation $P_1(x)$ to $f(x)$ on $[0, 1]$ and find the maximum error of $|P_1(x) - f(x)|$ on $[0, 1]$

   (c) Find the linear least squares approximation to f on $[0, 1]$.

2. The midpoint method for DE's is

$$\begin{aligned} w_0 &= \alpha \\ w_{n+1} &= w_n + hf\left(t_n + \frac{h}{2}, w_n + \frac{h}{2}f(t_n, w_n)\right). \end{aligned}$$

   Apply this method to the IVP, $y' = \lambda y$; $y(0) = 1$, with $\lambda < 0$ and find the constant c so that $|h\lambda| < c$ implies that $w_n \to 0$ as $n \to \infty$.

3. Derive this three-point formula for the second derivative.

$$f''(x) = \frac{1}{h^2}\left(f(x_0 - h) - 2f(x_0) + f(x_0 + h)\right) - \frac{h^2}{12}f^{(4)}(\eta)$$

   for some $\eta \in [x_0 - h, x_0 + h]$.

4. Let $f(x) = 2^x$ Let $x_0 = -1, x_1 = 0, x_2 = 1$. Find the total error bound for the degree two interpolating polynomial $p_2(x)$ with these nodes and f on the interval $[-1, 1]$, i.e. derive a K with

$$\max_{t \in [-1,1]} |f(t) - p_2(t)| < K.$$

5. Let $g \in C^2([a, b])$ and $p \in (a, b)$ with $g(p) = p, g'(p) = 0, g''(p) \neq 0$.

   (a) Show there is an $\epsilon > 0$ so that for all $x \in [p - \epsilon, p + \epsilon]$, we have $g^n(x) \to p$ as $n \to \infty$.

   (b) With ϵ as in part(a), show that for all $x \in [p-\epsilon, p+\epsilon]$, we have $|g(x)-p| \leq M|x-p|^2$ where $M = \max\{|g''(x)| \colon |x - p| \leq \epsilon\}/2$.