Logic PhD Exam, January 2023.

Solve 5 problems of the following; at least one from each section.

A. Set Theory.

1. Let X be an uncountable set. Let $f\colon X \to [X]^{<\aleph_0}$ be a function. Show that there are distinct points $x_0, x_1 \in X$ such that $x_0 \notin f(x_1)$ and $x_1 \notin f(x_0)$.
2. Let P be a forcing and κ be a regular cardinal larger than $|P|$. Show that $P \Vdash \check{\kappa}$ is a cardinal.
3. What is the perfect set theorem in descriptive set theory? State it and prove it.

B. Computability.

1. Choose one of the standard definitions of a recursive function, and show that the increasing enumeration of the Fibonacci numbers (defined by $f(0) = 0, f(1) = 1$ and $f(n+2) = f(n) + f(n+1)$) is a recursive function.
2. Provide an example of sets $A, B \subseteq \mathbb{N}$ such that A is Turing reducible to B but not many-one reducible to the complement of B.
3. Prove that a simple c.e. set exists, where a set $A \subseteq \mathbb{N}$ is simple if the complement of A has no infinite c.e. subset.

C. Model theory.

1. Let $\mathcal{L}$ be a language containing two binary relational symbols E_0, E_1. Let T be the theory stating that E_0, E_1 are equivalence relations and $E_0 \subseteq E_1$. Let $\mathcal{F}$ be the class of finite models of T. Show that $\mathcal{F}$ is a Fraissé class and provide a simple description of its limit.
2. State the Loś theorem and prove it.
3. Find a complete theory with more than one countably infinite model up to isomorphism. Provide two nonisomorphic countable models of the theory.