# Logic PhD Exam, August 2022

Solve 5 of the following problems; at least one from each section.

## A. Set Theory.

1. Let $X$ be an uncountable set and let $F : X \to X$ be a function. Show that there are distinct elements $x, y \in X$ such that $x \neq F(y)$ and $y \neq F(x)$.

2. Let $P$ be a forcing. Show that there is an uncountable cardinal $\kappa$ such that $P \Vdash \check{\kappa}$ is a cardinal.

3. Provide an example of a subset of a Polish space which is analytic and not Borel. Provide a proof of these two properties of your set.

## B. Computability.

1. Give an example of a subset of $\mathbb{N}$ that is $\Pi^0_1$ and not $\Sigma^0_1$ (and prove that your set has these two properties).

2. Prove that there exist incomparable Turing degrees.

3. Prove that there exist disjoint computably enumerable sets $A, B \subseteq \mathbb{N}$ such that there is no computable set $C$ which contains $A$ and is disjoint from $B$.

## C. Model Theory.

1. Let $\mathcal{L}$ be a language comprised of a single binary relation symbol. Give an example of a countably infinite $\mathcal{L}$-structure $\mathcal{M}$ such that

   (i) $\mathrm{Th}(\mathcal{M})$ is $\aleph_0$-categorical, and

   (ii) $\mathrm{Th}(\mathcal{M})$ does not have quantifier elimination.

Make sure to prove both statements (i) and (ii).

2. Let $\mathcal{L}$ be a language comprised of two unary relation symbols $G$ and $B$, and a binary relation symbol $R$. Let $\mathcal{K}$ be the class of all finite graphs with a bipartition, i.e., $\mathcal{K}$ consists of all finite $\mathcal{L}$-structures $\mathcal{M} = (M, G^{\mathcal{M}}, B^{\mathcal{M}}, R^{\mathcal{M}})$ such that

   (i) $R^{\mathcal{M}}$ is a symmetric and irreflexive relation on $M$,

   (ii) $G^{\mathcal{M}}$ and $B^{\mathcal{M}}$ form a partition of $M$,

   (iii) if $(x, y) \in R^{\mathcal{M}}$ then exactly one of $x, y$ belongs $G^{\mathcal{M}}$ and the other belongs to $B^{\mathcal{M}}$.

Prove or disprove: $\mathcal{K}$ is a Fraïssé class.

3. Let $\mathcal{L} = \{R\}$ be the language of graphs (i.e., $R$ is a binary relation symbol). Let $\mathcal{K}_0$ be the class of all connected graphs, and let $\mathcal{K}_1$ be the class of all disconnected graphs. Prove or disprove each of the following: (i) $\mathcal{K}_0$ is an elementary class; (ii) $\mathcal{K}_1$ is an elementary class.