Logic PhD Exam, May 2022.

Solve 5 problems of the following; at least one from each section.

A. Set Theory.

1. Sketch any proof of consistency of the Continuum Hypothesis with the axioms of ZFC.
2. What is a generic extension of a transitive model of ZFC? Provide a definition.
3. Show that every Polish space is a continuous image of the Baire space.

B. Computability.

1. Define many-one reducibility and many-one equivalence. Prove that there exist two non-computable, computably enumerable sets which are not many-one equivalent.
2. Prove Rice's Theorem: If P is a nonempty proper subset of the set of all partial computable functions, then the set of all indices of functions belonging to P is not computable.
3. Define the Turing degrees and the Turing jump operator. State the Friedberg Jump Inversion Theorem and outline its proof.
4. State and prove Kleene's Fixed Point Theorem (a.k.a. Recursion Theorem).

C. Model theory.

1. State both the upward and downward Löwenheim-Skolem Theorems. Prove either one of them.
2. Define what an ultraproduct of structures is and state Łoś's Theorem. Use Łoś's Theorem to give a proof of the Compactness Theorem.
3. Let $\mathcal{L}$ be a language consisting of one unary function symbol f. Consider the class K of all finite $\mathcal{L}$-structures in which f is a bijection. Prove that K is a Fraïssé class and describe its Fraïssé limit M. Prove or disprove: $\mathrm{Th}(M)$ is $\aleph_0$-categorical.
4. Let $\mathcal{L}$ be a language, let n be a number, let M be an $\mathcal{L}$-structure, and let A be a subset of M. Give the definition of an n-type of M over A. Define the Stone topology on the space S of complete n-types of M over A and prove that S equipped with this topology is a compact space.