Complex Analysis, PhD Examination, May 2023

Answer FIVE (out of eight) questions in detail, explaining your work in a precise and logical way. State carefully any results used without proof.

(1) Let $S = \{z \in \mathbb{C} : |z| < 2, \quad |z-1| > 1\}$ and $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Find an explicit analytic bijection $f : \mathbb{D} \to S$.

(2) Fix $\lambda > 1$. Show that the equation $\lambda - z - e^{-z} = 0$ has exactly one solution in the right half plane, $\mathbb{H} = \{z \in \mathbb{C} : \operatorname{real} z > 0\}$.

(3) Let A be an $n \times n$ matrix and $p(z) = \det(zI_n - A)$. Show $p(A) = 0$, where I_n is the $n \times n$ identity matrix.

(4) Show $e^z - z^2 = 0$ has infinitely many solutions.

(5) Find all solutions to $f^n + g^n = 1$ for positive integers $n \geq 2$ and entire functions f and g.

(6) Show $\prod_{n=2}^{\infty}(1 - \frac{1}{n^2}) = \frac{1}{2}$.

(7) Suppose $\Omega \subseteq \mathbb{C}$ is a region (non-empty, open and connected), (f_n) is a locally bounded sequence of analytic functions $f_n : \Omega \to \mathbb{C}$ and $S \subseteq \Omega$ has an accumulation point in Ω. Show, if $f : \Omega \to \mathbb{C}$ is an analytic function and $(f_n(s))$ converges to $f(s)$ for each $s \in S$, then (f_n) converges to f in $H(\Omega)$ (uniformly on compact sets).

(8) Let ζ denote the Riemann zeta function and σ the divisor function. Thus, for positive integers, $\sigma(k)$ is the number of divisors of k. Show, if $\operatorname{real} z > 1$, then

$$\zeta(z)^2 = \sum_{k=1}^{\infty} \frac{\sigma(k)}{k^z}.$$