Combinatorics Exam - May 2018

1. Consider the set T_n of (not rooted) trees with $n \geq 3$ labeled leaves for which each interior (non-leaf) vertex has degree 3.

   (a) Prove that each tree in T_n has exactly $2n - 3$ edges.

   (b) Prove that the number of trees in T_n is $(2n-5)!! = 1 \cdot 3 \cdot 5 \cdot 7 \cdots (2n-7) \cdot (2n-5)$.

2. The famous Dilworth Theorem for a finite poset P states that the minimum number of chains in any partition of P into chains is equal to the maximum number of elements in an antichain of P.

(a) Prove that if P has size at least $rs + 1$, then there is either a chain of size r or an antichain of size s.

(b) There are many proofs of the following Erdős-Szekeres theorem: For any sequence $a_1, a_2, \ldots, a_{n^2+1}$ of integers, there is a subsequence of length $n + 1$ that is monotone. Prove the Erdös-Sekeres theorem using Dilworth's theorem , i.e., part (a).

3. Prove that

$$\sum_{i=0}^{n} (-1)^i \binom{n}{i} i^k = \begin{cases} 0 & \text{if } \ 0 \le k < n \\ (-1)^n \, n! & \text{if } \ k = n. \end{cases}$$

What is the sum if $k > n$?

Hint: Count surjective mappings from $[k]$ to $[n]$.

4. (a) Prove that a planar graph of girth (minimum cycle length) at least 6 has a vertex of degree 2.

(b) The Four Color Theorem states that any planar graph is 4-colorable. Grotzsch proved that any triangle-free planar graph is 3-colorable. Without using his result, prove that a planar graph of girth at least 6 is 3-colorable.

Recall: The *girth* of a graph is the length of the shortest cycle (and ∞ if the graph has no cycles).

5. How many strings can be formed using the alphabet $\{A, B, C, D, E\}$ if

(a) the letter A occurs an odd number of times

(b) the letters A and B are both used an odd number of times.

6. Let $f(n)$ be the number of ways to tile a $1 \times n$ path with 1×2 tiles that are red, blue, or green, and 1×1 tiles that are yellow, orange, black, or white.

   (a) Find an explicit formula for $f(n)$.

   (b) On average, how many 1×1 tiles will a $1 \times n$ path contain?

7. Choose a derangement p of length n uniformly at random. Recall that a derangement is a permutation with no 1-cycles. On average, how many 2-cycles does p contain?

8. Let C be a binary code of length n and minimum distance $d \geq 2e + 1$. Prove the following two bounds (the Hamming bound and Singleton bound).

$$|C| \leq 2^n \Big/ \sum_{i=0}^{e} \binom{n}{i}$$

$$|C| \leq 2^{n-d+1}$$