Combinatorics Exam - January 2018

1. Let $D(n)$ be the number of derangements of length n. Let $F(n)$ be the number of permutations of length n that have exactly one fixed point. Determine (with proof) the number $D(n) - F(n)$.

2. Prove: If n and m are the number of vertices and edges in a simple planar graph, then $m \leq 3n - 6$.

   Prove: For a simple graph G with $n \geq 11$, at most one of G or its complement $\overline{G}$ is planar. (Recall that $\overline{G}$ has the same vertex set as G and, for all vertices u, v, we have that $\{u, v\}$ is an edge of $\overline{G}$ if and only if $\{u, v\}$ is not an edge of G.)

3. How many ways are there to seat n married couples at a straight table (only one side of the table) so that no woman sits next to her husband?

   How many ways are there to seat n married couples at a straight table so that men and women alternate and no woman sits next to her husband?

4. Let B be a rectangular 3-dimensional box for which the width is equal to the length, but the depth is not equal to the length. How many ways are there to color the vertices of B using colors red and blue, where two colorings are considered the same if one can be obtained from the other by an orientation preserving symmetry of B.

5. Consider a directed graph obtained by putting a direction on each edge of the complete graph. Such a digraph is called a *tournament*. Prove that a tournament has a Hamiltonian path. Is it true that every tournament has a Hamiltonian cycle? Explain?

6. Let $n \geq 3$. Let h_n be the number of permutations of length n in which the number of inversions is divisible by three. Find an explicit formula for h_n.

7. Prove that the language $\mathcal{L} = \{ww : w \in \{\mathsf{a}, \mathsf{b}\}^*\}$ is not regular.

8. Suppose $\mathcal{C}$ and $\mathcal{D}$ are permutation classes. Prove that $\mathcal{C} \cup \mathcal{D}$ is a permutation class, and describe (with an algorithm, perhaps) how to compute its basis.