UF Combinatorics PhD Exam — August 2023

1. Prove that for all positive integers n, the identity

$$n\binom{2n-1}{n-1} = \sum_{k=1}^{n} k\binom{n}{k}^2$$

holds.

2. Let B_n be the number of compositions of n in which the first part is odd. Find a closed-form, explicit formula (no summation signs) for $B_n + B_{n+1}$.

3. How many ways are there to choose a permutation of length n and color each of its cycles red, blue, or green? Give a closed-form, explicit formula (no summation signs) in terms of n.

4. Let w_n be the number of words of length n over the alphabet $\{A, B, C\}$ in which a B is not immediately followed by a C. Find the explicit form of the ordinary generating function of the sequence $w_0, w_1, w_2, \ldots$.

5. (a) For which nonnegative integers is the Catalan number C_n odd?
   (b) Let $a_0, a_1, a_2, \cdots$ be an infinite sequence that is polynomially recursive. Let b_i be the remainder of a_i modulo 2. Does it follow that the b_i form a polynomially recursive sequence?

6. Prove that an r-regular graph of girth (length of the shortest cycle) four has at least $2r$ vertices, with equality only for the complete bipartite graph $K_{r,r}$.

7. Let a and b be positive integers. We say that a is a *unitary divisor* of b if a divides b and a and b/a are relatively prime to each other. Let P be the poset of all positive integers in which $a \leq b$ if a is a unitary divisor of b. Describe the Möbius function of P. That is, provide a formula for $\mu_P(a, b)$.

8. Prove that every graph with m edges contains a bipartite subgraph with at least $m/2$ edges. (One approach is to compute the expected number of edges in a random bipartite subgraph, but there are also non-probabilistic proofs.)