Combinatorics Ph.D. exam

1. Let $a_1, a_2, \cdots, a_{10}$ be positive integers not exceeding 100. Prove that there are disjoint, nonempty subsets S and T of [10] so that

$$\sum_{i \in S} a_i = \sum_{j \in T} a_j.$$

2. Prove that the number of permutations of length $n + 1$ that have exactly two cycles is equal to the number of all cycles in all $n!$ permutations of length n. (For instance, if $n = 3$, then both of these numbers are equal to 11.)

3. Let P be a convex polyhedron whose faces are all either a-gons or b-gons, and whose vertices are each incident to three edges. Let p_a, p_b, and n respectively denote the number of a-gonal faces, b-gonal faces, and vertices of P. Prove that $p_a(6 - a) + p_b(6 - b) = 12$.

4. A city has three high schools, each of them attended by n students. Each student knows exactly $n + 1$ students who attend a high school *different from his.* Prove that we can choose three students, one from each school, so that each of them knows the other two.

5. Let L_n be the total number of leaves in all binary plane trees with n vertices. These are trees in which all non-leaf vertices have one or two children, and every child is a left child or a right child, even if it is an only child. So $L_1 = 1$, $L_2 = 2$, and $L_3 = 6$. Find an explicit formula for the numbers L_n.

6. Let $c : S \to \{0,1\}^*$ be a prefix-free code in which b_i codewords have length i. Prove that $\sum_i \frac{b_i}{2^i} \leq 1$.

7. Let T be a rooted non-plane 2-tree (that is, every non-leaf vertex has two children) with n leaves. For instance, for $n = 4$, there are two such trees, one in which each leaf is at edge-distance two from the root, and one in which the edge-distances of the leaves from the root are 1, 2, 3, and 3. Let $\text{sym}(T)$ be the number of non-leaf vertices v of T such that the two children of v are the roots of identical subtrees.

   (a) How many automorphisms does T have?

   (b) How many different ways are there to bijectively label the **leaves** of T with the numbers $1, 2, \cdots, n$?

   (c) Find an explicit formula for $\sum_T \frac{1}{|Aut(T)|}$, where the sum is taken over all rooted non-plane 2-trees with n leaves. For example, for $n = 4$, the two trees mentioned above have, respectively, eight and two automorphisms, so the requested sum is $\frac{1}{8} + \frac{1}{2} = \frac{5}{8}$.

   Your answers to parts (a) and (b) can contain $\text{sym}(T)$, but your answer to part (c) should not.

8. Let k be a fixed positive integer, and let $F_k(z) = \sum_{n\geq k} c(n,k)z^n/n!$, where $c(n,k)$ is the signless Stirling number of the first kind.

(a) Find $F_k(z)$ in an explicit form.

(b) Is $F_k(z)$ an algebraic power series over $\mathbf{C}$?

(c) Is $F_k(z)$ a d-finite power series over $\mathbf{C}$?