Combinatorics Exam

1. Prove that if every vertex of a graph G has degree at least 3, then G has a cycle with a chord. (A *chord* is an edge joining two non-adjacent vertices on the cycle.)

2. Let $\mathcal{T}$ be the set of (not rooted) trees with labeled leaves for which each vertex has degree 1 or 3, and let $\mathcal{T}_n \subseteq \mathcal{T}$ be the subset with exactly n leaves. Show that $T \in \mathcal{T}_n$ has exactly $2n-3$ edges and that $|\mathcal{T}_n| = (2n-5)!! = 1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-5)$.

3. Let Π_n be the set of permutations of $[n]$ such that $\pi(i+1) \leq \pi(i)+1$ for $i \in [n-1]$. Find $|\Pi_n|$.

4. Recall that Kőnig's theorem states that in a bipartite graph G, the maximum size of a matching is equal to the minimum size of an edge cover. Prove Kőnig's theorem from Hall's marriage theorem.

5. Let $(L, \leq)$ be a finite distributive lattice and $\mathrm{Irr}(L)$ be the set of join-irreducibles in L. For $t \in L$, define $K_t = \{p \in \mathrm{Irr}(L) : p \leq t\}$. Show
$$t = \bigvee_{p \in K_t} p.$$

6. Let $\mathcal{B}$ be the set of words in the alphabet $\{a, b, c\}$ so that the number of a's is even and the number of c's is odd. If b_n is the number of words in $\mathcal{B}$ with n letters, find
$$B(x) = \sum_{n \geq 0} b_n x^n.$$

7. Let Λ be the set of integer partitions. Prove the Cauchy identity
$$\prod_{i,j \geq 1} (1 - x_i y_j)^{-1} = \sum_{\lambda \in \Lambda} m_\lambda(x) h_\lambda(y)$$
where $m_\lambda(x)$ is the monomial symmetric function in the variables $x_1, x_2, \ldots$ and $h_\lambda(y)$ is the complete homogenous symmetric function in the variables $y_1, y_2 \ldots$.

8. Prove that every permutation of size $n > r \cdot s$ contains either an increasing subsequence of length r or a decreasing subsequence of length s.