# Combinatorics Exam

1. Let $a_n$ be the number of ways to select a permutation of length $n$, and then color each of its cycles red, blue, or green. Find an explicit formula for $a_n$.

2. Let $A$ be the graph obtained from $K_n$ by deleting an edge. Find a closed formula (so no summation signs) for the number of spanning trees of $A$.

3. Prove bijectively that the number of graphs with vertex set $[n]$ for which all vertices have even degree is $2^{\binom{n-1}{2}}$.

4. Prove that for every $n$, there is a tournament $T$ on $n$ vertices with at least $n!/2^{n-1}$ Hamiltonian paths.

5. Let $\mathbb{P} = \{1, 2, \dots\}$ and $(\mathbb{P}, |)$ be the division poset (so $m|n$ if $n/m \in \mathbb{P}$). Find a formula for the Möbius function $\mu$ and prove that it is correct.
   (Hint: decompose intervals in $(\mathbb{P}, |)$ as products of chains.)

6. Let $\mathcal{A}$ be the class of compositions into odd parts, where the size of $\alpha = (a_1, \dots, a_n) \in \mathcal{A}$ is $|\alpha| = a_1 + \cdots + a_n$. Find the associated ordinary generating function $A(x) = \sum_{n\geq 0} a_n x^n$ and compute its growth rate $\limsup a_n^{1/n}$.

7. Let $\mathcal{P}$ be the class of integer partitions. Give a bijective proof of the Cauchy identity

$$\prod_{i,j\geq 1} (1 - x_i y_j)^{-1} = \sum_{\lambda \in \mathcal{P}} s_\lambda(x_1, x_2, \dots) s_\lambda(y_1, y_2, \dots)$$

   where $s_\lambda$ is the Schur function of shape $\lambda$.

8. Let $\mathcal{A}$ and $\mathcal{B}$ be labeled combinatorial classes with size functions $|\cdot|_A$ and $|\cdot|_B$, respectively. Define an admissible construction $\star$ so that $\mathcal{C} = \mathcal{A} \star \mathcal{B}$ satisfies the exponential generating function identity $C_E(x) = A_E(x) \cdot B_E(x)$.