Combinatorics Exam - May 2017

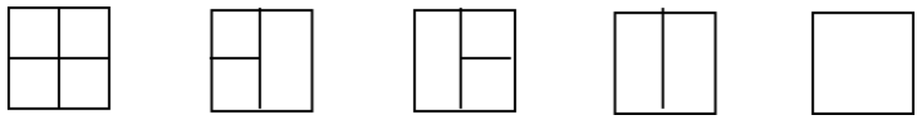
1. Give a combinatorial proof that
$$k\binom{n}{k} = n\binom{n-1}{k-1}.$$

2. You can use results from your course to prove:

(a) If a finite partially order set has $mn + 1$ elements, then there is either a chain of size $m + 1$ or an antichain of size $n + 1$.

(b) If G is a regular bipartite graph of degree at least 2, then G has a spanning subgraph that is the union of vertex disjoint cycles.

3. Let $a(n)$ be the number of ways to tile a $2 \times n$ rectangle with 1×1 squares and $2 \times k$ rectangles for any integer $k \geq 1$, where a 2×1 rectangle must be vertical (i.e. no 1×2 rectangle). For instance, $a(2) = 5$, given by



Set $a(0) = 1$, and find a simple expression for the generating function

$$A(x) = \sum_{n \geq 0} a(n)\, x^n.$$

4. Let C be a linear binary error correcting code. Prove that either every word in C has even weight or exactly half the words in C have even weight.

   Is a similar statement true for ternary codes?

5. Let $\mathcal{L}$ be a regular language. Prove that the language

$$\mathcal{L}_2 = \{w \, : \, ww \in \mathcal{L}\}$$

is regular.

6. Prove that the number of $k \cdots 21$-avoiding permutations is at most $(k-1)^{2n}$.

7. Let $c(n, k)$ be the number of permutations of length n with k cycles. Let a_n be the total number of cycles in all permutations of length n.

   Prove that

$$c(n+1, 2) = a_n.$$

8. Let b_n be the number of permutations of length n whose first ascent is in an even position. (For the decreasing permutation of length n, we say that its first ascent is in position n.)

   Find an explicit formula for b_n. Your formula can contain one summation sign.