PhD Analysis Exam, May 2018

Do six of eight. Answer each problem on a separate sheet of paper. Write solutions in a neat and logical fashion, giving complete reasons for all steps.

(1) Suppose $(X,\mathscr{M})$ and $(Y,\mathscr{N})$ are measurable spaces, $\mathscr{E}\subset\mathscr{N}$ and $f:X\to Y$. Show, if the σ-algebra generated by $\mathscr{E}$ contains $\mathscr{N}$ and $f^{-1}(E)\in\mathscr{M}$ for all $E\in\mathscr{E}$, then f is $\mathscr{M}-\mathscr{N}$ measurable.

(2) Do two. In each case, give a brief justification for your answer.

(a) Give an example, if possible, of a measurable set $C\subset\mathbb{R}$ of positive Lebesgue measure that contains no (non-trivial) interval.

(b) Give an example, if possible, of a measure space $(X,\mathscr{M},\mu)$ and a sequence of measurable functions $f_n:X\to\mathbb{C}$ that converge in measure, but not pointwise a.e.

(c) Give an example, if possible, of measure spaces $(X,\mathscr{M},\mu)$ and $(Y,\mathscr{N},\nu)$ and a function $f:X\times Y\to[0,\infty)$ that is measurable with respect to the product measure σ-algebra $\mathscr{M}\otimes\mathscr{N}$, but for which

$$\int_X\int_Y f(x,y)\,d\nu d\mu\neq\int_Y\int_X f(x,y)\,d\mu d\nu.$$

(3) Let $(X,\mathscr{M},\mu)$ be a σ-finite measure space (μ a positive measure). Suppose $\mathscr{N}$ is a sub-σ-algebra of $\mathscr{M}$ and $\nu=\mu|_{\mathscr{N}}$ is σ-finite. Given $f\in L^1(\mu)$, show, by considering the mapping $\rho:\mathscr{N}\to[0,\infty)$ defined by $\rho(E)=\int_E f\,d\mu$, that there exists an $\mathscr{N}$ measurable g such that $g\in L^1(\nu)$ and

$$\int_E f\,d\mu=\int_E g\,d\nu$$

for all $E\in\mathscr{N}$. Determine g in the case $(X,\mathscr{M},\mu)$ is the interval $[0,1]$ with Lebesgue measure and $\mathscr{N}=\{\varnothing,[0,\frac{1}{2}),[\frac{1}{2},1],[0,1]\}$.

(4) Show $f:[0,\infty)\to\mathbb{R}$ defined by $f(x)=(1+x)^{-2}$ is integrable (with respect to Lebesgue measure on $[0,\infty)$). Determine

$$\lim_{n\to\infty}\int_0^\infty\frac{\sin(\frac{x}{n})}{(1+(\frac{x}{n}))^n}\,dx.$$

(5) Prove, if X is a Banach space and M is a finite dimensional subspace of X, then there is a closed subspace N of X such that $M\cap N=(0)$ and for each $x\in X$ there exist $m\in M$ and $n\in N$ such that $x=m+n$.

(6) Let $(X, \mathscr{M}, \mu)$ be a measure space and suppose $1 \leq q < p < \infty$. Show, if $L^p(\mu) \subset L^q(\mu)$, then there is a constant κ such that if $E \in \mathscr{M}$ and $\mu(E) < \infty$, then $\mu(E) \leq \kappa$.

(7) Suppose $E \subset \mathbb{R}$ is (Lebesgue) measurable with positive measure. Show

$$E - E := \{x - y : x, y \in E\}$$

contains an open interval about 0.

(8) Recall c_{00} is the vector subspace of $\ell^\infty(\mathbb{N})$ consisting of sequences $a = (a_n)$ that are eventually zero (meaning there is an N, depending on a, such that $a_n = 0$ for $n > N$). Is there a norm on c_{00} that makes c_{00} a Banach space?