PhD Analysis Examination
May 2024

Answer each question on a separate sheet of paper. Write solutions in a neat and logical fashion, giving complete reasons for all steps.

Attempt SIX problems.

1. Consider Lebesgue measure on the real line $\mathbb{R}$. Give an example for each of the following, if possible. If not possible, give a brief explanation.

   A sequence f_n in $L^1(\mathbb{R})$ converging to an f in $L^1(\mathbb{R})$...

   a) ...in the L^1 norm but not in measure,

   b) ...in measure but not in the L^1 norm,

   c) ...in the L^1 norm but not a.e.,

   d) ...a.e. but not in measure.

2. State Tonelli's theorem, and give an example to show that the σ-finiteness hypothesis is necessary.

3. For σ-finte measures μ and ν, define what it means for μ to be *absolutely continuous* with respect to ν, and *singular* with respect to ν. State the Lebesgue-Radon-Nikodym theorem.

4. Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space. Suppose that g is a measurable function with the property that $gf \in L^2$ for all $f \in L^2$. Prove that $g \in L^\infty$.

5. Fix $1 < p < \infty$, let μ be a σ-finite measure, and let (f_n) be a sequence in $L^p(\mu)$. Suppose that there exists a function $f \in L^p(\mu)$ such that

   $$\int f_n g \, d\mu \to \int f g \, d\mu$$

   for every $g \in L^q(\mu)$, where $\frac{1}{p} + \frac{1}{q} = 1$.

   a) Prove that $\sup_n\{\|f_n\|_p\} < +\infty$.

   b) Prove that $\|f\|_p \leq \limsup \|f_n\|_p$.

   c) Give an example to show that strict inequality can hold in part (b).

6. Let $\mathcal{X}$ be a normed vector space. Say $x_n \to x$ *weakly* if $f(x_n) \to f(x)$ for all $f \in \mathcal{X}^*$. Prove that if $\mathcal{M}$ is a norm closed subspace of $\mathcal{X}$, and (x_n) is a sequence in $\mathcal{M}$ converging weakly to x, then $x \in \mathcal{M}$.

7. Let μ be a finite signed Borel measure on $[0,1]$. Suppose that

$$\int_0^1 \sin(2\pi n x)\, d\mu(x) = 0 \quad \text{and} \quad \int_0^1 \cos(2\pi n x)\, d\mu(x) = 0$$

for all $n = 0, 1, 2, \ldots$. Prove that $\mu = 0$. Does it suffice to assume only the vanishing of the sine integrals? Prove, or give a counterexample.

8. Short answer.

   a) Give a brief explanaation of how the Fourier transform is defined on $L^2(\mathbb{R})$.

   b) Sketch a proof that the Banach space $L^1(\mathbb{R})$ is not reflexive.

   c) Explain why the norm in $\ell^1(\mathbb{N})$ cannot be given by an inner product.