PhD Analysis Examination
January 2024

Answer each question on a separate sheet of paper. Write solutions in a neat and logical fashion, giving complete reasons for all steps.

Attempt SIX problems.

1. Let $(X, \mathscr{M}, \mu)$ be a finite measure space. Let (f_n) be a sequence of measurable functions and suppose that $\sum_n \int |f_n| \, d\mu < \infty$. Consider the series $\sum_n f_n(x)$. Does this series converge...

   a) pointwise a.e.?

   b) in measure?

   c) in the L^1 norm?

   Prove your claims.

2. State Tonelli's theorem, and give an example to show that the σ-finiteness hypothesis is necessary.

3. a) Let ν be a signed measure on $(X, \mathscr{M})$. Define what it means for a set to be *positive, negative* or *null* for ν, and state the Hahn decomposition theorem.

   b) Prove that if ν is a signed measure, then there exist unique positive measures ν^+, ν^- so that $\nu = \nu^+ - \nu^-$ and $\nu^+ \perp \nu^-$.

4. Let $\varphi_n : \mathbb{R} \to \mathbb{C}$ be a sequence of Lebesgue measurable functions and suppose that there is a number M such that $\|\varphi_n\|_\infty \leq M$ for all n. Prove that if

$$\int_a^b \varphi_n(t) \, dt \to 0$$

   for every interval $[a, b] \subset \mathbb{R}$, then

$$\int_{-\infty}^{\infty} \varphi_n(t) f(t) \, dt \to 0$$

   for every $f \in L^1(\mathbb{R})$.

5. Let μ and ν be finite, positive measures defined on the same measurable space $(X, \mathcal{M})$, and suppose that $\nu \ll \mu$.

   a) Prove that if $L^1(\mu) \subset L^1(\nu)$, then the inclusion map $\iota : L^1(\mu) \hookrightarrow L^1(\nu)$ is necessarily bounded.

   b) Prove that if $L^1(\mu) \subset L^1(\nu)$ then $\frac{d\nu}{d\mu} \in L^\infty(\mu)$.

6. Let $\mathcal{X}, \mathcal{Y}$ be Banach spaces and $T : \mathcal{X} \to \mathcal{Y}$ a linear map. Prove that if $f \circ T \in \mathcal{X}^*$ for all $f \in \mathcal{Y}^*$, then T is bounded.

7. Short answer.

   a) Explain why there is no norm on c_{00} (the space of finitely nonzero sequences) which makes it into a Banach space.

   b) Explain why there is no bounded, surjective linear map from c_0 onto ℓ^∞.

   c) Explain why the norm in $L^1(\mathbb{R})$ cannot be given by an inner product.

8. Let H be a Hilbert space with orthonormal basis $\{e_k\}_{k=1}^\infty$. Let (h_n) be a sequence in H and let $h \in H$. Prove that the following statements are equivalent:

   1) $\langle h_n, g \rangle \to \langle h, g \rangle$ for all $g \in H$.

   2) $\langle h_n, e_k \rangle \to \langle h, e_k \rangle$ for all $k = 1, 2, \ldots,$ AND $\sup_n \|h_n\| < \infty$.