PhD Analysis Examination
August 2023

Answer each question on a separate sheet of paper. Write solutions in a neat and logical fashion, giving complete reasons for all steps.

You must attempt SIX problems.

1. State Tonelli's theorem, and give an example to show that the σ-finiteness hypothesis is necessary.

2. Give an example of each of the following, if possible. (If not possible, give a brief explanation of why.)

   a) a sequence of functions converging in measure but not in L^1,

   b) a sequence converging almost uniformly but not essentially uniformly,

   c) a sequence converging in L^1 but not almost everywhere.

   (When giving examples, be clear about what measure space you are working in.)

3. Let $(X, \mathscr{M}, \mu)$ be a measure space. Prove that $L^2(\mu)$ is complete (in the L^2 norm).

4. Let $\mathcal{X}$ be a Banach space. Prove that every finite-dimensional subspace $\mathcal{M} \subset \mathcal{X}$ is closed.

5. a) State the Lebesgue-Radon-Nikodym theorem. b) State and prove a version of the chain rule for Radon-Nikodym derivatives.

6. Let $(X, \mathscr{M}, \mu)$ be a σ-finite measure space, let $1 \leq q < p < \infty$ and suppose that $L^q(\mu) \subseteq L^p(\mu)$. Prove that there exists a constant $c > 0$ such that, for every $E \in \mathscr{M}$, either $\mu(E) = 0$ or $\mu(E) \geq c$.

7. Let $\mathcal{H}$ be a Hilbert space and let $A : \mathcal{H} \to \mathcal{H}$ be a linear map (not assumed bounded). Suppose there exists another linear map $B : \mathcal{H} \to \mathcal{H}$ (not assumed bounded) so that $\langle Ax, y \rangle = \langle x, By \rangle$ for all $x, y \in H$. Prove that A and B are bounded.

8. a) State the Closed Graph Theorem and Banach Isomorphism theorem. b) Using the Banach Isomorphism theorem (or otherwise), prove the Closed Graph Theorem.