PhD Analysis Examination
January 2023

Answer each question on a separate sheet of paper. Write solutions in a neat and logical fashion, giving complete reasons for all steps.

Attempt SIX problems.

1. For any two of the following, either give an example or explain why no example exists.
   a) a sequence of Lebesgue measurable functions on $\mathbb{R}$ which converges in measure but not Lebesgue a.e.
   b) a closed set $E \subset [0,1]$ with positive Lebesgue measure, but which contains no open intervals
   c) a decreasing sequence of nonnegative measurable functions on $\mathbb{R}$ such that $\lim \int f_n \neq \int \lim f_n$

2. Prove that for any positive measure μ, the space $L^1(\mu)$ is complete in the L^1 norm.

3. Let (μ_n) be a sequence of finite signed measures on a measurable space $(X, \mathscr{M})$. Prove that there exists a finite, positive measure ν on $(X, \mathscr{M})$ such that $\mu_n \ll \nu$ for all n.

4. Suppose that $f \in L^1[0,1]$ and $\int_0^x f\, dm = 0$ for all $x \in [0,1]$. What can you conclude about f? (Prove your claim.)

5. Let $\mathcal{X}$ be a Banach space. Say that a sequence $(x_n) \subset \mathcal{X}$ converges weakly to $x \in \mathcal{X}$ if $f(x_n) \to f(x)$ for all $f \in \mathcal{X}^*$. Prove the following:
   a) If (x_n) converges weakly, then $\sup_n \|x_n\| < \infty$.
   b) Weak limits are unique. (That is, if $x_n \to x$ weakly and $x_n \to y$ weakly, then $x = y$.)

6. a) State the closed graph theorem. b) Let $\mathcal{X}$ be a Banach space with closed subspaces $\mathcal{Y}, \mathcal{Z} \subset \mathcal{X}$. Suppose that every $x \in \mathcal{X}$ admits a *unique* decomposition $x = y + z$ with $y \in \mathcal{Y}$, $z \in \mathcal{Z}$. Prove that there is a constant C so that $\|y\| \leq C\|x\|$ and $\|z\| \leq C\|x\|$.

7. Let H be a Hilbert space and $T : H \to H$ a bounded linear operator. Define the *adjoint operator* $T^* : H \to H$ and prove that it is bounded, with $\|T^*\| = \|T\|$.

8. (Short answer, you do not need to give a detailed proof, just a brief explanation.)
   a) Define the Fourier transform on $L^1(\mathbb{R})$ and sketch its extension to $L^2(\mathbb{R})$.
   b) Sketch a construction of a bounded linear functional λ on $L^\infty(\mathbb{R})$ which is *not* of the form $\lambda(f) = \int f(x)g(x)\, dx$ for any L^1 function g.
   c) Sketch a proof of the fact that there is no norm under which c_{00} is complete.