PhD Analysis Examination
August 2022

Answer each question on a separate sheet of paper. Write solutions in a neat and logical fashion, giving complete reasons for all steps.

You must attempt SIX problems.

1. Let $(X, \mathscr{M})$ be a measurable space and ν a signed measure. Define the *Hahn decomposition* of X and the *Jordan decomposition* of ν. Prove that the Jordan decomposition is unique.

2. Let μ, ν be positive measures defined on the same measure space. Define what it means for ν to be *absolutely continuous* with respect to μ, and state the Radon-Nikodym theorem.

3. State Tonelli's theorem, and give an example to show that the σ-finiteness hypothesis is necessary.

4. Give an example of each of the following, if possible. (If not possible, give a brief explanation of why.)

   a) a sequence of functions converging in measure but not in L^1,

   b) a sequence converging almost uniformly but not essentially uniformly,

   c) a sequence converging in L^1 but not almost everywhere.

   (When giving examples, be clear about what measure space you are working in.)

5. Let $(X, \mathscr{M}, \mu)$ be a σ-finite measure space. Prove that the simple functions which belong to $L^2(\mu)$ are dense in $L^2(\mu)$.

6. Let $\mathcal{X}$ be a Banach space and $\mathcal{N} \subset \mathcal{X}$ a closed subspace, and let $\pi : \mathcal{X} \to \mathcal{X}/\mathcal{N}$ be the quotient map. Show that if $f : \mathcal{X} \to \mathbb{F}$ is a bounded linear functional such that $f(n) = 0$ for all $n \in \mathcal{N}$, then there is a bounded linear functional $g : \mathcal{X}/\mathcal{N} \to \mathbb{F}$ such that $f = g \circ \pi$.

7. Let $\mathcal{X}$ be a reflexive Banach space, with dual space $\mathcal{X}^*$. Suppose $B : \mathcal{X} \times \mathcal{X}^* \to \mathbb{F}$ is a bilinear mapping, and there is a constant $C > 0$ so that

   $$|B(x, x^*)| \leq C\|x\|\|x^*\| \text{ for all } x \in \mathcal{X},\ x^* \in \mathcal{X}^*.$$

   Prove that there is a bounded linear operator $T : \mathcal{X} \to \mathcal{X}$ such that $B(x, x^*) = x^*(Tx)$ for all $x \in \mathcal{X}, x^* \in \mathcal{X}^*$. Must T be unique?

8. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function with compact support, and let $\phi : \mathbb{R} \to \mathbb{R}$ be a nonnegative L^1 function with $\int_{-\infty}^{\infty} \phi(t)\, dt = 1$. For each real $\lambda > 0$, consider the function

   $$f_\lambda(x) := \frac{1}{\lambda} \int_{-\infty}^{\infty} \phi\left(\frac{x-t}{\lambda}\right) f(t)\, dt.$$

   Prove that $f_\lambda \to f$ uniformly on $\mathbb{R}$ as $\lambda \to 0$.